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11. Exercise Sheet for Algorithms in Numerical Mathematics

Exercise 30: The Arnoldi method is applied to $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Show that:

(a) If $h_{k+1,k} = 0$, then $\mathcal{K}_k = \mathcal{K}_k(A, b)$ is an A-invariant subspace of $\mathbb{R}^n$, i.e., $A\mathcal{K}_k \subseteq \mathcal{K}_k$, and it holds that $\mathcal{K}_k = \mathcal{K}_{k+1} = \cdots = \mathcal{K}_n$.

(b) If k is the degree of the minimal polynomial of A, then there exists a $j \leq k$ such that $h_{j+1,j} = 0$.

Exercise 31: To solve $Ax = b$ with a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, the GMRES or FOM method is used. Let $\{v_1, \ldots, v_k\}$ be the Arnoldi basis for the starting vector b. Show that:

(a) When applying the Arnoldi method to the transformed problem $\hat{A}\hat{x} = \hat{b}$ with $\hat{A} = QAQ^T$ and $\hat{b} = Qb$, where Q is an orthogonal matrix, then for the vectors $\{\hat{v}_1, \ldots, \hat{v}_k\}$ of the new Arnoldi basis it holds that $\hat{v}_j = Qv_j$.

(b) Use this to show that GMRES and FOM for the transformed problem $\hat{A}\hat{x} = \hat{b}$ yield the solution $\hat{x} = Qx$.

Exercise 32: Show that the Lanczos and Arnoldi methods are invariant under shifts, i.e., if A is replaced by $A + \lambda I$ with $\lambda \in \mathbb{R}$, the Krylov bases V_k (and W_k for Lanczos) remain unchanged, and the Hessenberg matrices H_k (and $\tilde{T}_k$ for Lanczos) are transformed to $H_k + \lambda I$.

Exercise 33: Show that the residual of the QMR-method stagnates, i.e. $x_k^{\mathrm{QMR}} = x_{k-1}^{\mathrm{QMR}}$, exactly when the k-th BiCG iterate x_k^{BiCG} doesn't exist.

Hint: If one cannot augment the Krylov subspace further, they are to keep the last computed one.

Solutions are discussed on Tuesday 08.07.2025.
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.