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10. Exercise Sheet for Algorithms in Numerical Mathematics

Exercise 27: (Preconditioning)

For the preconditioned CG-method, the following is true for the initial matrix A and the preconditioning matrix B:

$$\gamma(v, B^{-1}v) \leq (v, Av) \leq \Gamma(v, B^{-1}v), \quad \text{for all } v \in \mathbb{R}^n,$$

where $\gamma, \Gamma > 0$. Show that it holds for the error after k steps

$$\|x_k - x\|_A \leq 2\left(\frac{\sqrt{\tilde{\kappa}} - 1}{\sqrt{\tilde{\kappa}} + 1}\right)^k \|x_0 - x\|_A, \quad \text{with } \tilde{\kappa} = \frac{\Gamma}{\gamma}.$$

Hint: According to the lecture, it suffices to show that with $B = CC^T$ and $\tilde{A} = C^T AC$ holds: $\mathrm{cond}_2(\tilde{A}) \leq \Gamma/\gamma$.

Exercise 28:

The linear equation system $Ax = b$ with symmetric and positive definite matrix A is to be solved with and without preconditioning. Let the condition number of A be 10.000, that of the preconditioned system 100.

Give upper bounds for the number of iterations that the method of steepest descent (without preconditioning) and the cg method (with and without preconditioning) need to reduce the error (measured in the A-norm) by a factor of 10^5?

Exercise 29: (Fletcher-Reeves)

With the cg-method of Fletcher-Reeves one can solve the one-dimensional minimization methods approximately until the termination condition

$$|g_{k+1}^T d_k| \leq \sigma g_k^T d_k, \quad \sigma \in \left(0, \frac{1}{2}\right]$$

is reached.

Show that then the procedure for each k is a descent procedure i.e., that the search direction $-d_k$ is a descent direction, i.e., that for small $\alpha > 0$ holds

$$f(x_k - \alpha d_k) < f(x_k).$$

Hint: Show via induction that for $d_0 = g_0$ it holds

$$\left|\frac{g_k^T d_k}{g_k^T g_k} - 1\right| \leq \sum_{j=0}^{k} \sigma^j - 1.$$

Which values can $g_k^T d_k$ be? Interpret $g_k^T d_k$ geometrically.

Programming Exercise 10: Implement the preconditioned cg-method for solving a linear equation system $Ax = b$ with symmetric positive definite matrix A. Use (and implement by yourself) the incomplete Cholesky decomposition as a preconditioner. Plot the error $\|Ax_k - b\|$ for all k. Test your code for the same test matrices as in programming exercise 9.

Solutions are discussed on Tuesday 01.07.2025.
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.