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8. Exercise Sheet for Algorithms in Numerical Mathematics

Exercise 23:

Let $V \leq W \leq \mathbb{R}^n$ and $V^\perp = \{w \in W : w^T A v = 0 \text{ for all } v \in V\}$ be the subspace of W conjugate to V ($A \in \mathbb{R}^{n\times n}$ symmetric positive definite). Show that for all $w \in W$ there exist uniquely defined $v \in V$ and $d \in V^\perp$ with

$$w = v + d,$$

where

$$v = \sum_{j=0}^{k-1} \frac{\langle Ad_j, w\rangle}{\langle Ad_j, d_j\rangle} d_j,$$

if $\{d_0, \dots, d_{k-1}\}$ is an A-orthogonal basis of V (i.e., $\langle Ad_i, d_j\rangle = 0$ for $i \neq j$).

Exercise 24:

How can the method of steepest descent be used to solve the linear equation system $Ax = b$ with symmetric positive definite matrix A? Show: In the norm $\|v\|_A = \sqrt{v^T A v}$, it holds for the error

$$\|x_k - x\|_A \leq \left(1 - \frac{1}{\mathrm{cond}_2(A)}\right)^{k/2} \|x_0 - x\|_A.$$

($\mathrm{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2 = \lambda_{\max}(A)/\lambda_{\min}(A)$). Hence the procedure converges (but very slowly if A is ill conditioned).

Note: Use (for example) the following structure for the proof.

$$\|x_k - x\|_A^2 = \ldots = d_k^T A^{-1} d_k = \ldots = d_k^T A^{-1} d_{k-1} = \ldots$$

$$= d_{k-1}^T A^{-1} d_{k-1} \left(1 - \frac{d_{k-1}^T d_{k-1}}{d_{k-1}^T A d_{k-1}} \frac{d_{k-1}^T d_{k-1}}{d_{k-1}^T A^{-1} d_{k-1}}\right) \leq \ldots = \|x_{k-1} - x\|_A^2 \left(1 - \frac{1}{\mathrm{cond}_2(A)}\right).$$

Programming Exercise 6:

(a) Let A be a $m \times n$ matrix with given singular value decomposition $A = U\Sigma V^T$, where $\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$ and Σ_r is non-singular.

Show that, the solution of the linear fitting problem

$$\|Ax - b\|_2 = \min \quad \text{with} \quad \|x\|_2 = \min$$

is given by

$$x = A^+ b, \quad \text{where} \quad A^+ = V\Sigma^+ U^T$$

with

$$\Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) Write a program that solves this in the proposed way. Test your code for at least one rectangular matrix A of your choice, i.e. $m > n$.
Hint. You may use the 'svd'-function of Matlab or Julia. You do not need to program the singular value decomposition by yourself.

Programming Exercise 7:

Write a program that uses the singular value decomposition (SVD) for image compressing of a black and white image. I.e. you have an image as an input and a compressed, memory reduced version of the same image as an output. Can you estimate the error in a certain norm? Compute different rank-r approximations of the black and white image for r = (50, 100, 200). Apply your code to the test picture you can find on the website. Hints:

(a) You can use `[U,S,V] =svd(A)` in Matlab or `using LinearAlgebra; U,S,Vt = svd(A)` in Julia.

(b) Use the gray function shown in Programming Exercise 2, so as to use only one matrix.

(c) Use the show function shown in Programming Exercise 2 to visualize your results too.

Solutions are discussed on Tuesday 17.06.2025.
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.