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4. Exercise Sheet for Algorithms in Numerical Mathematics

Exercise 11: For given $\alpha > 0$, let u_α be the solution of the minimization problem (R) from the lecture. Show (for $\alpha > 0$):

(a) $\alpha \mapsto \|a * u_\alpha - b\|_{L^2}$ increases monotonously,

(b) $\alpha \mapsto \|u_\alpha^{(p)}\|_{L^2}$ decreases monotonously.

Exercise 12: (Condition Number)

(a) Let λ be a simple zero of the characteristic polynomial of $A \in \mathbb{R}^{n\times n}$. Show that the condition number of the eigenvalue λ of A exists (i.e. $u^*v \neq 0$) and is invariant under unitary similarity transformations (i.e., that the eigenvalue λ of the matrix U^*AU with unitary matrix U, has the same condition number).

(b) Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable with pairwise different eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$ and left eigenvectors $u_1^*, \dots, u_n^*$. Let further $C \in \mathbb{R}^{n\times n}$ be arbitrary.

Show: The matrix $A + \varepsilon C$ has the eigenvectors

$$v_j(\varepsilon) = v_j + \varepsilon \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{1}{\lambda_j - \lambda_i} \frac{u_i^* C v_j}{u_i^* v_i} v_i + O(\varepsilon^2)$$

Note: Express $v_j'(0)$ as a linear combination of v_i. To determine the coefficients of v_i $(i \neq j)$, use that $u_i^* v_j = 0$ for $i \neq j$ (why?). Consider a suitably scaled $v_j(\varepsilon)$ to also get the coefficient of v_j as claimed.

Exercise 13: (Theorem of Gerschgorin)

a) Show: The union of all circular disks

$$K_i = \{\mu \in \mathbb{C} : \ |\mu - a_{i,i}| \leq \sum_{\substack{k=1\\ k\neq i}}^{n} |a_{i,k}|\}$$

contains all eigenvalues of the $n \times n$ matrix $A = (a_{i,j})$.
Hint: Consider the equation $Ax = \lambda x$ component-wise.

b) Draw all Gerschgorin circles of the matrix:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & 4 \\ 1 & 2 & 10 \end{pmatrix}.$$

How can one further restrict the set of possible eigenvalues?
Optional: You can draw the circles using Matlab or Julia.

Programming Exercise 3:

Write a program that adds a normally distributed perturbation to given data and then smoothes this data. Plot the data, the perpetuated data and the smoothed data graphically. Alternatively, try and understand the following Matlab program:

```
N=256;
x=(2*pi/N)*[0:N-1]';     % grid
f=sin(x)+0.2*sin(3*x)-0.2*cos(6*x); % undefined function

e=0.1*randn(N,1);        % perturbation normally distributed
% with scatter 0.1

b=f+e;                   % perturbed values in b

bb=fft(b);               % inverse FFT
n=[0:N/2-1 -N/2:-1]';
alpha=0.0001;            % regularization parameter
uu=bb./(1+alpha*n.^4);   % filter
u=ifft(uu);              % FFT, smoothed data in u
plot(x,[real(u),f,b]);   % plot f,u,b as functions of x
delta=norm(e)/sqrt(N)
d=norm(u-b)/sqrt(N)
```

Test your (or the above) program with several values of the regularization parameter α. Modify the program so that it also computes a smoothed derivative of the function and return the result.

Solutions are discussed on Tuesday 29.04.2025.
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.