Universität Tübingen
Mathematisches Institut
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11. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 32: Understand a simplex step using the following linear optimization task as an example

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 12 \\ x \geq 0 \\ x_1 + x_2 + x_3 = \min! \end{cases}$$

$x = (12, 0, 0)^T$ is chosen as the initial edge. Consider the following:

(a) How does the cost $x_1 + x_2 + x_3$ change as the second component of x grows?

(b) How does the cost $x_1 + x_2 + x_3$ change as the third component of x grows?

(c) Which component of x is increased in the first step of the simplex procedure and why?

(d) What does the new edge look like then?

(e) Why is this edge optimal?

Exercise 33: Determine an edge for the linear optimization task

$$\begin{cases} x_1 - x_2 - x_3 = 6 \\ x \geq 0 \\ x_1 + x_2 + x_3 = \min! \end{cases}$$

with the "phase I" of the simplex algorithm.

Exercise 34: Consider for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ with $n \geq m$ the problem

$$\begin{cases} Ax = b \; x \geq 0 \\ c^T x = \min! \end{cases}$$

with admissible set $D = \{y \in \mathbb{R}^n; Ay = b, y \geq 0\}$. Show: if $\operatorname{rang}(A) = m$, then there always exists an edge $x \in D$.
Hint: $x \in D$ is an edge if and only if x cannot be represented as a convex combination of two points y, $z \in D$, i.e. if

$$x = (1 - \lambda)y + \lambda z \text{ with } \lambda \in [0, 1],\, y,\, z \in D \quad \Rightarrow \quad x = y = z.$$

Further consider: a given $x \in \mathbb{R}^n$ with $x_i \geq 0$ for $i \in \mathcal{I}$ and $x_i = 0$ für $i \notin \mathcal{I}$, where $\mathcal{I} \subseteq \{1, \dots, n\}$ is an m-elementary subset, and $A_{\mathcal{I}} = (Ae_j)_{j \in \mathcal{I}}$ is an edge if $A_{\mathcal{I}}$ is regular. Then consider a vector $x \in D$ with minimal number of non-vanishing components and show that the reduced matrix $A_{\mathcal{I}}$ has full rank, where $\mathcal{I}$ are those indices whose associated x-component does not vanish. Finally, use $\operatorname{rank}(A) = m$.

Exercise 35: (Conditions for optimality)

For the linear optimization task $Ax = b$, $x \geq 0$, $c^T x$ minimal! let

$$L(x, y) := c^T x - y^T (Ax - b)$$

be the Lagrange function. Show for $x \geq 0$: x, y are optimal for the primal and dual problems, respectively, if and only if (x, y) is the saddle point of L, i.e.

$$\max_{v \in \mathbb{R}^m} L(x, v) = L(x, y) = \min_{u \in \mathbb{R}^n_+} L(u, y).$$

Note: Choose skillfully for the reverse directions v and u.

To get admitted to the exam you need at least 22 crosses
Solutions are discussed on Wednesday 19.07.2023.
Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.