Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

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10. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 28: (Preconditioning)

For the preconditioned cg method, the following is true for the initial matrix A and the preconditioning matrix B:

$$\gamma(v, B^{-1}v) \leq (v, Av) \leq \Gamma(v, B^{-1}v), \qquad \text{for all } v \in \mathbb{R}^n,$$

where $\gamma, \Gamma > 0$. Show that it holds for the error after k steps

$$\|x_k - x\|_A \leq 2\left(\frac{\sqrt{\tilde\kappa}-1}{\sqrt{\tilde\kappa}+1}\right)^k \|x_0 - x\|_A, \qquad \text{with } \tilde\kappa = \frac{\Gamma}{\gamma}.$$

Hint: According to the lecture, it suffices to show that with $B = CC^T$ and $\widetilde{A} = C^TAC$ holds: $\operatorname{cond}_2(\widetilde{A}) \leq \Gamma/\gamma$.

Exercise 29:

The linear equation system $Ax = b$ with symmetric and positive definite matrix A is to be solved with and without preconditioning. Let the condition number of A be 10.000, that of the preconditioned system 100.

Give upper bounds for the number of iterations that the method of steepest descent (without preconditioning) and the cg method (with and without preconditioning) need to reduce the error (measured in the A-norm) by a factor of 10^5?

Exercise 30: (Fletcher–Reeves)

With the cg-method of Fletcher-Reeves one can solve the one-dimensional minimization methods approximately until the termination condition

$$|g_{k+1}^T d_k| \leq \sigma g_k^T d_k, \qquad \sigma \in \left(0, \frac{1}{2}\right]$$

is reached.

Show that then the procedure for each k is a descent procedure i.e., that the search direction $-d_k$ is a descent direction, i.e., that for small $\alpha > 0$ holds

$$f(x_k - \alpha d_k) < f(x_k).$$

Hint: Show via induction that for $d_0 = g_0$ it holds

$$\left|\frac{g_k^T d_k}{g_k^T g_k} - 1\right| \leq \sum_{j=0}^{k} \sigma^j - 1.$$

Which values can $g_k^T d_k$ be? Interpret $g_k^T d_k$ geometrically.

Exercise 31: (Termination with Arnoldi)

The Arnoldi method is applied to $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Show:

(a) If $h_{k+1,k} = 0$, then the k-th Krylov-space K_k is a A-invariant subspace of $\mathbb{R}^n$, i.e. $AK_k \subseteq K_k$, and it holds $K_k = K_{k+1} = \ldots = K_N$.

(b) If k is the degree fo the minimal polynomial of A then there is a $j \leq k$, such that $h_{j+1,j} = 0$.

Solutions are discussed on Wednesday 12.07.2023.

Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.