Universität Tübingen
Mathematisches Institut
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9. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 26:

Show: For the cg-method it holds

$$\frac{(d_k, g_k)}{(Ad_k, d_k)} = \frac{(g_k, g_k)}{(Ad_k, d_k)}, \qquad \frac{(Ad_k, g_{k+1})}{(Ad_k, d_k)} = -\frac{(g_{k+1}, g_{k+1})}{(g_k, g_k)}.$$

Exercise 27:

Let the eigenvalues of A (symmetric and positiv definit) be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Show: With $\kappa' = \lambda_2/\lambda_n$ it holds for the error in the cg-method

$$\|x_k - x\|_A \leq 2 \left(\frac{\sqrt{\kappa'} - 1}{\sqrt{\kappa'} + 1} \right)^{k-1} \|x_0 - x\|_A \quad \text{for } k \geq 2.$$

If $\lambda_1 \gg \lambda_2$, this is far stronger as the similar estimate with $\kappa = \lambda_1/\lambda_n$ from the lecture.

Hint: $q_k(\lambda) = \widetilde{q}_{k-1}(\lambda) \cdot (\lambda_1 - \lambda)/\lambda_1$.

Programming exercise 8:

Implement the cg-method for solving a linear equation system $Ax = b$ with symmetric positive definite matrix A. Plot the error $||Ax_k - b||$ for all k. Then test your function using the following two matrices.
Matrix 1:

```
function A = MatrixGenerator(N)
A = -4*diag(ones(N^2,1)) - diag(ones(N*(N-1),1),N) - diag(ones(N*(N-1),1),-N);
for i=0:N-1
   for j=1:N-1
      A(j+i*N,j+1+i*N) = -1;
      A(j+1+i*N,j+i*N) = -1;
   end
end
```

Matrix 2:

```
A = sprandsym(N^2,0.2,0.1) + 2*diag(ones(N^2,1));
```

both for $N = 4, 20, 40$. Further choose $b = ones(N^2, 1)$.

Programming exercise 9:

Implement the preconditioned cg-method for solving a linear equation system $Ax = b$ with symmetric positive definite matrix A. Use (and implement by yourself) the incomplete Cholesky decomposition as a preconditioner. Plot the error $||Ax_k - b||$ for all k. Test your code for the same test matrices as in programming exercise 8.

Solutions are discussed on Wednesday 05.07.2023.
Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.