Universität Tübingen
Mathematisches Institut
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Tübingen, 21.06.2023

8. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 23:

Let A be a $m \times n$ matrix with given singular value decomposition $A = U\Sigma V^T$ with $\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$, Σ_r non singular.

Show that the solution of the linear fitting problem

$$\|Ax - b\|_2 = \min, \qquad \|x\|_2 = \min$$

is given by

$$x = A^+ b, \qquad \text{with } A^+ = V\Sigma^+ U^T \text{ and } \Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Exercise 24:

Let $V \le W \le \mathbb{R}^n$ and $V^\perp = \{w \in W : w^T A v = 0 \text{ for all } v \in V\}$ be the subspace of W conjugate to V ($A \in \mathbb{R}^{n\times n}$ symmetric positive definite). Show that to all $w \in W$ there exist uniquely defined $v \in V$ and $d \in V^\perp$ with

$$w = v + d,$$

where $v = \sum_{j=0}^{k-1} \frac{(Ad_j, w)}{(Ad_j, d_j)} d_j$, if $d_0, \ldots, d_{k-1}$ is an A-orthogonal basis of V (i.e. $(Ad_i, d_j) = 0$ for $i \neq j$).

Exercise 25: (Method of steepest descent)

How can the method of steepest descent be used to solve the linear equation system $Ax = b$ with symmetric positive definite matrix A?

Show: In the norm $\|v\|_A = \sqrt{v^T A v}$, it holds for the error

$$\|x_k - x\|_A \le \left(1 - \frac{1}{\mathrm{cond}_2(A)}\right)^{k/2} \|x_0 - x\|_A.$$

($\mathrm{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2 = \lambda_{\max}(A)/\lambda_{\min}(A)$.) Hence the procedure converges (but very slowly if A is ill conditioned).

Note: Use (for example) the following structure for the proof.

$$\begin{aligned} \|x_k - x\|_A^2 &= \ldots = d_k^T A^{-1} d_k = \ldots = d_k^T A^{-1} d_{k-1} = \ldots \\ &= d_{k-1}^T A^{-1} d_{k-1}\Big(1 - \frac{d_{k-1}^T d_{k-1}}{d_{k-1}^T A d_{k-1}} \frac{d_{k-1}^T d_{k-1}}{d_{k-1}^T A^{-1} d_{k-1}}\Big) \le \ldots = \|x_{k-1} - x\|_A^2 \Big(1 - \frac{1}{\mathrm{cond}_2(A)}\Big). \end{aligned}$$

Programming exercise 7:

Write a Matlab program that solves the problem from exercise 23 in the proposed way. Test your code for at least one rectangular matrix A of your choice, i.e. $m > n$.
Hint. You may use the 'svd'-function of Matlab. You do not need to program the singular value decomposition by yourself.

Solutions are discussed on Wednesday 28.06.2023.
Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.