Universität Tübingen
Mathematisches Institut
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Tübingen, 10.05.2021

4. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 11: (condition number)

(a) Let λ be a simple zero of the characteristic polynomial of $A \in \mathbb{R}^{n\times n}$. Show that the condition number of the eigenvalue λ of A exists (i.e. $u^*v \neq 0$) and is invariant under unitary similarity transformations (i.e., that the eigenvalue λ of the matrix U^*AU with unitär matrix U, has the same condition number).

(b) Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable with pairwise different eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$ and left eigenvectors $u_1^*, \ldots, u_n^*$. Let further $C \in \mathbb{R}^{n\times n}$ be arbitrary.

Show: The matrix $A + \varepsilon C$ has the eigenvectors

$$v_j(\varepsilon) = v_j + \varepsilon \sum_{i=1, i\neq j}^{n} \frac{1}{\lambda_j - \lambda_i} \frac{u_i^* C v_j}{u_i^* v_i} v_i + O(\varepsilon^2)$$

Note: Express $v_j'(0)$ as a linear combination of v_i. To determine the coefficients of v_i $(i \neq j)$, use that $u_i^* v_j = 0$ für $i \neq j$ (why?). Consider a suitably scaled $v_j(\varepsilon)$ to also get the coefficient of v_j as claimed.

Exercise 12:

Show: If B is a normal and A is any $n \times n$ matrix, then for every eigenvalue λ of A there is an eigenvalue μ of B with

$$|\lambda - \mu| \leq ||A - B||_2$$

Hint: Show first: If λ is not an eigenvalue of B, then:

$$||(\lambda I - B)^{-1}||_2 = \frac{1}{\min\limits_{\mu \in \lambda(B)} |\lambda - \mu|}.$$

Then consider for the eigenvector x of A belonging to λ the vector $(A - B)x$.

Exercise 13: (Theorem of Gerschgorin)

a) Show: The union of all circular disks

$$K_i = \{\mu \in \mathbb{C} : \ |\mu - a_{i,i}| \leq \sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{i,k}|\}$$

contains all eigenvalues of the $n \times n$ matrix $A = (a_{i,j})$.

Hint: Consider the equation $Ax = \lambda x$ component-wise.

b) Draw all Gerschgorin circles of the matrix.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & 4 \\ 1 & 2 & 10 \end{pmatrix}.$$

How can one further restrict the set of possible eigenvalues?

Exercise 14:

Calculate the eigenvalues of the $n \times n$ matrix $\tilde{A} = A + \epsilon C$ with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \vdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}, \quad C = \hat{e}_n \hat{e}_1^T.$$

What do you get for $n = 8$ and $\epsilon = 10^{-8}$?

Solutions are discussed on Wednesday 17.05.2023.
Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.