Universität Tübingen
Mathematisches Institut
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3. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 8: (Fourier series)

Let $c = (c_n)_{n=-\infty}^{\infty}$ be an absolute summable sequence of complex numbers. Show:

(a)
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \hat{c}(\theta)e^{-in\theta}d\theta \quad , \qquad \text{where} \quad \hat{c}(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

(b)
$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{c}(\theta)|^2 d\theta$$

(c) If the sequence $d = (d_n)_{n=-\infty}^{\infty}$ is also absolutely summable, the convolution $c * d$, defined by $(c * d)_n = \sum_{j=-\infty}^{\infty} c_{n-j}d_j$, is also defined. Further, it holds
$$\widehat{c * d} = \hat{c} \cdot \hat{d}.$$

Exercise 9:

Let
$$u_N(x) = \sum_{n=-N/2}^{N/2-1} \hat{u}_N(n)e^{inx}, \qquad b_N(x) = \sum_{n=-N/2}^{N/2-1} \hat{b}_N(n)e^{inx}$$
trigonometric interpolation polynomials to 2π-periodic continuous functions $u(x)$ and $b(x)$. Then for $x_j = j\frac{2\pi}{N}$ it holds
$$\|u_N - b_N\|_{L^2}^2 = \frac{1}{N}\sum_{j=0}^{N-1} |u_N(x_j) - b(x_j)|^2.$$

Note: First apply the Parselval's equation from §4 and then the equation from §1.

Exercise 10:

For given $\alpha > 0$, let u_α be the solution of the minimization problem (R) from the lecture. Show (for $\alpha > 0$):

(a) $\alpha \mapsto \|a * u_\alpha - b\|_{L^2}$ increases monotonous,

(b) $\alpha \mapsto \|u_\alpha^{(p)}\|_{L^2}$ decreases monotonous.

Programming exercise 2:

Write a program that adds a normally distributed perturbation to given data and then smoothes this data. Plot the data, the perpetuated data and the smoothed data graphically. Alternatively, try and understand the following Matlab program:

```
N=256;
x=(2*pi/N)*[0:N-1]'; % grid
f=sin(x)+0.2*sin(3*x)-0.2*cos(6*x); % undefined function

e=0.1*randn(N,1); % pertubation normally distributed
% with scatter 0.1

b=f+e; % perturbed values in b

bb=fft(b); % inverse FFT
n=[0:N/2-1 -N/2:-1]';
alpha=0.0001; % regularization parameter
uu=bb./(1+alpha*n.^4); % filter
u=ifft(uu); % FFT, smoothed data in u
plot(x,[real(u),f,b]); % plot f,u,b as functions of x
delta=norm(e)/sqrt(N)
d=norm(u-b)/sqrt(N)
```

Test your (or the above) program with several values of the regularization parameter α. Modify the program so that it also computes a smoothed derivative of the function and return the result.

Solutions are discussed on Wednesday 10.05.2023.
Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.
Note: There will be no lecture on Wednesday 10.05.2023.