Universität Tübingen
Mathematisches Institut
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2. Exercise sheet for Algorithmen der Numerischen Mathematik

Exercise 5: (sorting problem)

A file of $N = 2^L$ names is to be sorted alphabetically. Give an algorithm that does this in $\mathcal{O}(N\ \log N)$ operations. We assume that names can be compared in $\mathcal{O}(1)$.
Note: Divide et impera!

Exercise 6: Let f be continuous and 2π-periodic with absolutely summable Fourier coefficients $(\widehat{f}(n))_{n\in\mathbb{Z}}$. Their approximation by the midpoint rule is

$$\widetilde{\widehat{f}_N}(n) = \frac{1}{N}\sum_{j=0}^{N-1} f(t_j)e^{-int_j} \qquad \text{with} \quad t_j = \frac{2j+1}{2}\cdot\frac{2\pi}{N}.$$

Show the aliasing formula

$$\widetilde{\widehat{f}_N}(n) = \sum_{l=-\infty}^{\infty} (-1)^l \widehat{f}(n+lN).$$

Exercise 7: (Chebyshev interpolation)

The function $g : [-1,1] \to \mathbb{R}$ can be expressed by Chebyshev polynomials

$$g(x) = \frac{1}{2}\gamma_0 + \sum_{k=1}^{\infty} \gamma_k T_k(x)$$

with absolute summable coefficients (γ_k). Let

$$p(x) = \frac{1}{2}\tilde{\gamma}_0 + \sum_{k=1}^{n} \tilde{\gamma}_k T_k(x)$$

be the Chebyshev interpolation polynomial to g of degree n, where you know from Numerics I that

$$\tilde{\gamma}_k = \frac{2}{n+1}\sum_{j=0}^{n} g(cos(t_j))\cos(kt_j), \qquad t_j = \frac{2j+1}{n+1}\cdot\frac{\pi}{2}.$$

Show:

$$\tilde{\gamma}_k = \sum_{l=-\infty}^{\infty} (-1)^l \gamma_{k+2(n+1)l}.$$

Hint: It holds $\gamma_{-k} = \gamma_k$, für all $k \in \mathbb{Z}$.

Programming exercise 1: Implement the fast Fourier transform (without using `fft` and `ifft`). You may assume that the length of the input vector is a power of two.
Note: Implement the fast Fourier transform recursively (i.e., your function recalls itself).

Solutions are discussed on Wednesday 03.05.2023.
Contact person: Dominik Sulz - when you have questions just come to my office (C3P16) or write me an email.