Universität Tübingen
Mathematisches Institut
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Tübingen, November 4, 2026

3. Exercise Sheet for Numerics

Exercise 9: Determine the interpolation polynomial $p(x)$ of degree two for a function f given the following data points:

x_j	t	$t+h/2$	$t+h$
y_j	$f(t)$	$f(t+h/2)$	$f(t+h)$

$t \in \mathbb{R},\ h > 0.$

Furthermore, show that by integrating this polynomial from t to $t+h$, one obtains Simpson's rule.

Exercise 10: Given the following table of values:

x_i	-1	0	1	3
y_i	8	3	4	8

.

(a) Interpolate the table of values using Newton's interpolation formula.

(b) Let $(x_4, y_4) = (2, 1)$. What is the Newton interpolation polynomial when including the point (x_4, y_4)?

(c) Using the Lagrange interpolation formula, determine the unique polynomial of degree three that passes through the above data points.

(d) Compare the computational effort of evaluating the interpolation polynomial at a point x in the Newton and Lagrange forms.

Exercise 11: Given the interpolation polynomial $p(x)$ of $f(x)$ at the interpolation points $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$, where f is $(n+1)$-times continuously differentiable.

(a) Show that $p(x)$ converges to the n-th Taylor polynomial of $f(x)$ at x_0 as $h \to 0$.

(b) Determine the form of the interpolation error $f(x) - p(x)$ as $h \to 0$.

Exercise 12: Show that for a polynomial p of degree n, a useful and easily computable estimate of $\max_{x\in[-1,1]} |p(x)|$ is given by $\max_{k=0,\ldots,n} |p(x_k)|$, where x_k are the roots of the Chebyshev polynomial T_{n+1}. Find a constant C_n that is as small as possible, such that

$$\max_{x\in[-1,1]} |p(x)| \le C_n \cdot \max_{k=0,\ldots,n} |p(x_k)|.$$

Programming Exercise 2: (Adaptive Numerical Integration)

Write a function `adaptint`(f, a, b, tol) that calculates the integral

$$\int_a^b f(x)\,dx$$

using Simpson's rule, with the absolute error being less than tol. The basic interval should be adaptively subdivided through recursive calls to `adaptint`. Use the midpoint rule to estimate the error. Write a script `adaptint_test` to calculate an approximation of the integral

$$\int_0^4 x^2 e^{-5x}\, dx$$

Hint: The function `adaptint(f,a,b,tol)` calculates approximations of $\int_a^b f(x)\, dx$ using both Simpson's rule and the midpoint rule. If $|\text{Simpson} - \text{Midpoint}| > \text{tol}$, the interval $[a, b]$ is halved, and the result is given by

$$\text{adaptint}(f, a, (a+b)/2, \text{tol}/2) + \text{adaptint}(f, (a+b)/2, b, \text{tol}/2).$$

Otherwise, the result is accepted as Simpson's approximation.

Solutions are discussed on November 11, 2026.
Programming Exercises can be submitted until November 17, 2026, at 23:59 at `https://na.uni-tuebingen.de/ex/num1_ws2627/`.