Universität Tübingen
Mathematisches Institut
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2. Exercise Sheet for Numerics

Exercise 5: Prove

$$\int_0^1 K_p(t)\,dt = \frac{1}{p!}\left(\frac{1}{p+1} - \sum_{i=1}^{s} b_i c_p^i\right).$$

Exercise 6:

(a) Show that for any positive, continuous function ω on $[a,b]$, the following defines an inner product on the space of continuous, real-valued functions:

$$(f,g) := \int_a^b f(x)g(x)\omega(x)\,dx.$$

(b) (Rodrigues' Formula) Show that the orthogonal polynomials p_k with respect to the weight function ω on the interval $[a,b]$ satisfy

$$p_k(x) = C_k \frac{1}{\omega(x)}\frac{d^k}{dx^k}\left[\omega(x)(x-a)^k(b-x)^k\right], \quad C_k \in \mathbb{R},$$

if the right side is a polynomial of degree k.

Hint: Prove that the polynomial defined above is orthogonal to all polynomials of degree $\leq k-1$ using integration by parts.

Exercise 7: Given the trapezoidal rule (TR) and the midpoint rule (MR) for numerical approximation of an integral, determine the nodes and weights of the quadrature formula

$$\int_a^b f(x)\,dx \approx \alpha \cdot \mathrm{TR} + \beta \cdot \mathrm{MR}$$

in terms of α and β. For which choice of the parameters will the order be maximal? Justify your answer.

Exercise 8: The Legendre polynomials P_k are normalized by the condition $P_k(1) = 1$.

(a) Using Exercise 6, show that:

$$P_k(x) = \frac{(-1)^k}{2^k\,k!}\frac{d^k}{dx^k}\left[(1-x^2)^k\right].$$

(b) Show the recursion for the Legendre polynomials:

$$P_{k+1}(x) = \frac{2k+1}{k+1}xP_k(x) - \frac{k}{k+1}P_{k-1}(x).$$

Solutions are discussed on November 4, 2026.