Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

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1. Exercise Sheet for Numerics

Exercise 1: (Landau-Notation)
For (real) functions f and g, we write $f = \mathcal{O}(g)$ for $x \to a$ (where $a \in \mathbb{R} \cup \{\pm\infty\}$) if there exists a neighborhood U of a and a constant $C \in \mathbb{R}$ such that

$$|f(x)| \leq C|g(x)| \quad \text{for all } x \in U$$

(or more precisely, if $\limsup_{x \to a} \frac{|f(x)|}{|g(x)|} < \infty$). Intuitively, this means that the function f does not grow faster than the function g in a neighborhood of a.

Given the functions

$$x^3, \log(x), 2^x, x^2, x^3 + 1000x^2, e^x,$$

compare the growth of these functions as $x \to \infty$ and $x \to 0$ using the $\mathcal{O}$-notation described above.

Exercise 2: Approximate the value of the integral

$$\int_0^2 x^2 e^{-5x}\, dx$$

first by applying Simpson's Rule twice, and then by applying it four times on equally spaced intervals. What can be inferred about the error when using Simpson's Rule four times instead of twice?

Exercise 3: Let the nodes $c_1 = 0$ and $c_3 = 1$ of a quadrature formula for $s = 3$ be given. Determine the node c_2 as well as the weights b_1, b_2, and b_3 so that the order of the quadrature formula is maximized. What is the order of your quadrature formula?

Exercise 4: Show the following error estimate for the midpoint rule:

$$\left| \int_{x_0}^{x_0+h} f(x)\, dx - hf\left(x_0 + \frac{h}{2}\right) \right| \leq \frac{h^3}{24} \max_{x \in [x_0, x_0+h]} |f''(x)|.$$

Programming Exercise 1:

(a) Write a function `quadrature(f, a, b, N, rule)` that expects the following arguments (in the order given): a function $f : [a, b] \to \mathbb{R}$; real numbers a and b; an integer N; and a string rule, which can take the value `"rectangle"`, `"trapezoid"`, or `"simpson"`. Depending on the value of rule, the integral

$$\int_a^b f(x)\, dx$$

should be approximated using the rectangle rule, the trapezoidal rule, and Simpson's rule. N specifies the number of (equally spaced) subintervals.

(b) Determine the exact value of the integral

$$\int_0^2 \cos(x) e^{\sin(x)}\, dx.$$

Write a script `quad_plot` that performs the following task: Calculate the approximation of the integral for all methods mentioned in (a) with $N = 2, 4, 8, 16, 32, 64$. Let h be the length of the respective subintervals. Generate a plot for each quadrature formula, using the MATLAB command `loglog`, or in Julia with `Plots` as shown in the Introduction to Julia course, which shows the logarithm of the error as a function of $\log(h)$. In the same plot, draw the function h for the rectangle rule, h^2 for the trapezoidal rule, and h^4 for Simpson's rule.

What do you observe? Can you explain the behavior?

Notes: The `loglog` command works like the `plot` command, but it plots $\log(f(\log(x)))$ against $\log(x)$. With `figure(k)`, you can instruct MATLAB to draw the following plot in the k-th figure. In Julia you can do the following:

```
using Plots
x = abs.(rand(10))
y = abs.(rand(10))
p = plot(x, y, xscale=:log10, yscale=:log10, label="plot_1")
plot!(p, x, x, label="plot_2")
```

Solutions are discussed on October 28, 2026
Programming Exercises can be submitted until November 3, 2026, at 23:59 at `https://na.uni-tuebingen.de/ex/num1_ws2627/`