Exercise Sheet: Lecture 3 – Brownian Motion and Itô Calculus

– Solutions –

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Lecture 3: Advanced Exercises – Solutions

This sheet provides detailed solutions and explanations to the key concepts of Brownian motion, stochastic calculus, the Itô integral, and stochastic differential equations.

1 Exercise 1: Simulation of Brownian Motion

A standard Brownian motion $(W_t)_{t\geq 0}$ has independent, stationary increments with $W_0 = 0$ and $W_{t+h} - W_t \sim N(0, h)$. To simulate a path on $[0, T]$ with step size h, we proceed as follows:

- Set $N = \lceil T/h \rceil$ and times $t_k = kh$ for $k = 0, 1, \ldots, N$, with $t_N = T$.
- Initialize $W(0) = 0$.
- For $k = 1$ to N, generate independent standard normal $Z_k \sim N(0, 1)$ and set

  $$W(t_k) = W(t_{k-1}) + \sqrt{h}\, Z_k,$$

  since $W(t_k) - W(t_{k-1}) \sim N(0, h)$ by stationarity of increments.
- The vector $(W(t_0), W(t_1), \ldots, W(t_N))$ is then a discrete approximation of a Wiener path.

In pseudocode (e.g. Python-like) one could write:

```
W[0] = 0
for k in 1...N:
Z = standard_normal()
W[k] = W[k-1] + sqrt(h) * Z
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This generates one sample path of Brownian motion on $[0, T]$ with increments $W(t_k) - W(t_{k-1}) \sim N(0, h)$.

2 Exercise 2: Total Variation

The *total variation* of a (continuous) function f on $[a, b]$ is defined by

$$V_{a,b}(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. We show that for a Brownian path $t \mapsto W_t$, $V_{a,b}(W)$ is infinite almost surely.

Fix an interval $[a, b]$ and consider a uniform partition with $t_i = a + i(b - a)/N$. Let $\Delta W_i = W(t_i) - W(t_{i-1})$. Then one uses the identity

$$\sum_{i=1}^{N} \Delta W_i^2 \;\leq\; \left(\max_i |\Delta W_i|\right) \sum_{i=1}^{N} |\Delta W_i|.$$

As the mesh of the partition goes to 0, continuity of W implies $\max_i |\Delta W_i| \to 0$ almost surely. On the other hand, the left side $\sum \Delta W_i^2$ has expected value $b - a$ (independent of N) and in fact converges to $b - a$ in L^2 (see Exercise 3 below). Hence for large N, with high probability $\sum \Delta W_i^2$ is bounded away from 0, forcing $\sum |\Delta W_i|$ to grow without bound as $N \to \infty$. Formally, as $N \to \infty$ we have $\sum |\Delta W_i| \to \infty$ almost surely. Thus the total variation is infinite with probability one.

For example, a more detailed argument yields that $V_{a,b}(W) = \infty$ almost surely. Intuitively, Brownian paths are nowhere differentiable and oscillate infinitely often, so the accumulated absolute increments diverge.

3 Exercise 3: Quadratic Variation

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be any partition of $[0, T]$ with mesh $\max_i(t_i - t_{i-1}) \to 0$. Define the quadratic variation sum

$$Q_N = \sum_{i=1}^{N} (W_{t_i} - W_{t_{i-1}})^2.$$

We show $Q_N \to T$ in L^2, i.e. $E[(Q_N - T)^2] \to 0$.

First, by independent increments and $E[(W_{t_i} - W_{t_{i-1}})^2] = t_i - t_{i-1}$ (since $\mathrm{Var}(W_{t_i} - W_{t_{i-1}}) = t_i - t_{i-1}$), we have

$$E[Q_N] = \sum_{i=1}^{N} E[(W_{t_i} - W_{t_{i-1}})^2] = \sum_{i=1}^{N} (t_i - t_{i-1}) = T.$$

Thus Q_N is an unbiased estimator of T. Next, compute the variance:

$$\mathrm{Var}(Q_N) = E[(Q_N - E[Q_N])^2] = E\left[\left(\sum_{i=1}^{N} (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right)^2\right].$$

Because the increments are independent and $\mathrm{Var}((W_{t_i} - W_{t_{i-1}})^2) = 2(t_i - t_{i-1})^2$ (using that $W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})$ has fourth moment $3(t_i - t_{i-1})^2$), one finds

$$\mathrm{Var}(Q_N) = \sum_{i=1}^{N} 2(t_i - t_{i-1})^2 \leq 2 \max_i (t_i - t_{i-1}) \sum_{i=1}^{N} (t_i - t_{i-1}) = 2T \max_i (t_i - t_{i-1}).$$

Since the mesh $\max_i (t_i - t_{i-1}) \to 0$, it follows that $\mathrm{Var}(Q_N) \to 0$. Hence $E[(Q_N - T)^2] = \mathrm{Var}(Q_N) \to 0$. This implies $Q_N \to T$ in L^2 (and also in probability).

In summary, as the partition is refined, the sum of squared increments converges in mean square to T, the length of the interval. This shows Brownian motion has *quadratic variation* T over $[0, T]$.

4 Exercise 4: Covariance Function

For a standard Brownian motion with $W_0 = 0$, we compute the covariance $E[W_s W_t]$ for $s, t \geq 0$. Without loss of generality assume $s \leq t$. Then $W_t = W_s + (W_t - W_s)$, where $W_t - W_s$ is independent of $\sigma(W_u : u \leq s)$ and has mean 0. Using linearity and independence,

$$E[W_s W_t] = E\big[W_s(W_s + (W_t - W_s))\big] = E[W_s^2] + E[W_s]\, E[W_t - W_s] = E[W_s^2] + 0.$$

Since $W_s \sim N(0, s)$, $E[W_s^2] = \mathrm{Var}(W_s) = s$. Hence $E[W_s W_t] = s$. By symmetry, if $t \leq s$ one finds $E[W_s W_t] = t$. In either case,

$$E[W_s W_t] = \min(s, t)\,.$$

This is the well-known covariance function of Brownian motion.

5 Exercise 5: Markov Property

A stochastic process is *Markov* if the conditional distribution of the future, given the present state, depends only on that present state and not on the past history. For Brownian motion, we fix $0 \le s < t$ and let $\mathcal{F}_s$ be the σ-algebra generated by $\{W_u : 0 \le u \le s\}$ (the "past up to time s"). The Markov property in this context means $W_t - W_s$ is independent of $\mathcal{F}_s$. But this is true by the independent increment property of Brownian motion: disjoint increments are independent. More precisely, $\{W_{s+u} - W_s : u \ge 0\}$ is a Brownian motion independent of $\mathcal{F}_s$, hence the increment $W_t - W_s$ is independent of what happened before time s. Equivalently, for any bounded measurable function g,

$$E\big[g(W_t) \mid \mathcal{F}_s\big] = E\big[g(W_s + (W_t - W_s)) \mid \mathcal{F}_s\big] = E[g(W_s + Z)]\Big|_{Z \sim N(0,t-s)},$$

which depends only on W_s, not on earlier history. This establishes the Markov property of Brownian motion.

6 Exercise 6: Distribution of Increments

Let $0 \le s < t$. By the stationary increments property, the distribution of $W_t - W_s$ depends only on $t - s$ and is the same as that of $W_{t-s} - W_0 = W_{t-s}$. Since $W_{t-s} \sim N(0, t-s)$ by definition of standard Brownian motion, it follows that

$$W_t - W_s \sim N(0,\, t - s).$$

Moreover, if we have two non-overlapping intervals $[s_1, t_1]$ and $[s_2, t_2]$ with $t_1 \le s_2$, then $(W_{t_1} - W_{s_1})$ and $(W_{t_2} - W_{s_2})$ are increments over disjoint time sets and so are independent by the independent increment property. Thus non-overlapping increments are independent. In summary, Brownian motion has Gaussian increments $N(0, t-s)$ over $[s, t]$ and these increments are independent over disjoint intervals.

7 Exercise 7: Itô Isometry for Elementary Functions

Let ϕ be an *elementary (simple) adapted process* on $[0, T]$, meaning there is a partition $0 = t_0 < t_1 < \cdots < t_n = T$ and $\phi(t) = \phi_k$ for $t_{k-1} < t \le t_k$, where each ϕ_k is $\mathcal{F}_{t_{k-1}}$-measurable. The Itô integral is defined by

$$\int_0^T \phi(t)\, dW_t = \sum_{k=1}^{n} \phi_k \big(W_{t_k} - W_{t_{k-1}}\big).$$

We compute its second moment. Since $E[W_{t_k} - W_{t_{k-1}}] = 0$ and different increments are independent, the cross terms drop out. Specifically,

$$\begin{aligned} E\Big[\Big(\int_0^T \phi(t)\,dW_t\Big)^2\Big] &= E\Big[\Big(\sum_{k=1}^n \phi_k \Delta W_k\Big)^2\Big] \\ &= \sum_{k=1}^n E[\phi_k^2 (\Delta W_k)^2] + 2\sum_{k<\ell} E[\phi_k \phi_\ell \Delta W_k \Delta W_\ell]. \end{aligned}$$

For $k < \ell$, $\Delta W_k = W_{t_k} - W_{t_{k-1}}$ and $\Delta W_\ell = W_{t_\ell} - W_{t_{\ell-1}}$ are independent, and ϕ_k is $\mathcal{F}_{t_{k-1}}$-measurable (hence independent of ΔW_ℓ). Thus

$$E[\phi_k \phi_\ell \Delta W_k \Delta W_\ell] = E[\phi_k \phi_\ell] E(\Delta W_k) E(\Delta W_\ell) = 0.$$

So only the diagonal terms remain. Also $\Delta W_k \sim N(0, t_k - t_{k-1})$ independent of ϕ_k, so

$$E[\phi_k^2 (\Delta W_k)^2] = E\big[\phi_k^2 E((\Delta W_k)^2 \mid \phi_k)\big] = E\big[\phi_k^2 (t_k - t_{k-1})\big].$$

Hence

$$E\Big[\Big(\int_0^T \phi(t)\,dW_t\Big)^2\Big] = \sum_{k=1}^n E[\phi_k^2](t_k - t_{k-1}) = E\Big[\sum_{k=1}^n \phi_k^2 (t_k - t_{k-1})\Big] = E\Big[\int_0^T \phi(t)^2\,dt\Big].$$

This is the Itô isometry for elementary ϕ:

$$E\Big[\Big(\int_0^T \phi\,dW\Big)^2\Big] = E\Big[\int_0^T \phi(t)^2\,dt\Big].$$

8 Exercise 8: Itô Integral $\int_0^t W(s)\,dW(s)$

We compute the Itô integral of $W(s)$ with respect to $W(s)$. By definition, for a partition $0 = t_0 < \cdots < t_n = t$, the Riemann-sum approximation is

$$I_n = \sum_{k=1}^n W(t_{k-1})\big(W(t_k) - W(t_{k-1})\big).$$

Observe the algebraic identity for each increment:

$$W(t_k)^2 - W(t_{k-1})^2 = (W(t_k) - W(t_{k-1}))(W(t_k) + W(t_{k-1})).$$

Rearrange to express $W(t_{k-1})(W(t_k) - W(t_{k-1}))$:

$$W(t_{k-1})(W(t_k) - W(t_{k-1})) = \frac{1}{2}\left(W(t_k)^2 - W(t_{k-1})^2 - (W(t_k) - W(t_{k-1}))^2\right).$$

Summing over $k = 1, \dots, n$ gives

$$I_n = \frac{1}{2}\left(W(t_n)^2 - W(t_0)^2 - \sum_{k=1}^{n}(W(t_k) - W(t_{k-1}))^2\right).$$

Since $W(0) = 0$, $W(t_n) = W(t)$, and by Exercise 3 $\sum_{k=1}^{n}(W(t_k) - W(t_{k-1}))^2 \to t$ in mean square as the mesh $\to 0$. Hence in the limit

$$\int_0^t W(s)\, dW(s) = \lim_{n\to\infty} I_n = \frac{1}{2}\left(W(t)^2 - t\right).$$

Therefore

$$\int_0^t W(s)\, dW(s) = \frac{1}{2}W(t)^2 - \frac{1}{2}t.$$

This is the classical result: the Itô integral of W against itself equals $(W(t)^2 - t)/2$.

9 Exercise 9: Itô vs Riemann–Stieltjes Integral

Let $v : [0, t] \to \mathbb{R}$ be a deterministic C^1 function with $v(0) = 0$. Then the Riemann–Stieltjes integral $\int_0^t v(s)\, dv(s)$ coincides with the ordinary integral $\int_0^t v(s)v'(s)\, ds$, since $dv(s) = v'(s)ds$. Hence by the fundamental theorem of calculus,

$$\int_0^t v(s)\, dv(s) = \int_0^t v(s)v'(s)\, ds = \frac{1}{2}v(t)^2 - \frac{1}{2}v(0)^2 = \frac{1}{2}v(t)^2.$$

In contrast, for Brownian motion we found in Exercise 8 that

$$\int_0^t W(s)\, dW(s) = \frac{1}{2}W(t)^2 - \frac{1}{2}t,$$

which has the extra drift term $-\frac{1}{2}t$. The reason is that Brownian motion has nonzero quadratic variation: heuristically $(dW)^2 = dt$, whereas for a smooth function $(dv)^2 = 0$. In other words, the second-order Itô correction term appears only in the stochastic case. The extra $-\frac{1}{2}t$ arises from the term $\frac{1}{2}F_{xx}(W)\,(dW)^2$ in Itô's formula, reflecting the randomness of W (no such term appears for deterministic v).

10 Exercise 10: Martingale Property

A stochastic process $(X_t)_{t\geq 0}$ is a martingale with respect to a filtration $(\mathcal{F}_t)$ if $E[|X_t|] < \infty$ and

$$E[X_t \mid \mathcal{F}_s] = X_s, \quad \text{for all } s < t.$$

For Brownian motion, take $\mathcal{F}_s = \sigma\{W_u : 0 \le u \le s\}$. We check

$$E[W_t \mid \mathcal{F}_s].$$

Using $W_t = W_s + (W_t - W_s)$, and the fact that W_s is $\mathcal{F}_s$-measurable while the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and has mean 0, we obtain:

$$E[W_t \mid \mathcal{F}_s] = E[\,W_s + (W_t - W_s) \mid \mathcal{F}_s] = W_s + E[W_t - W_s] = W_s.$$

Thus $E[W_t \mid \mathcal{F}_s] = W_s$ for all $s \le t$, and W_t has finite variance, so (W_t) is a martingale with respect to its natural filtration.

11 Exercise 11: Adapted Processes

A stochastic process $X = (X_t)_{t \ge 0}$ is said to be *adapted* to a filtration $(\mathcal{F}_t)_{t \ge 0}$ if for each time t, the random variable X_t is $\mathcal{F}_t$-measurable. Informally, this means that X_t depends only on information available up to time t (it is non-anticipative).

- **Example of adapted process:** Any process with its own natural filtration is adapted to that filtration. For instance, Brownian motion W_t is adapted to its natural filtration $\mathcal{F}_t = \sigma\{W_s : 0 \le s \le t\}$, since W_t is by definition measurable w.r.t. $\mathcal{F}_t$. More concretely, the process $X_t = W_t^2$ is also adapted, because its value at time t is determined by W_t (which is known at time t).
- **Non-example:** Consider a process $Y_t = W_T$ for a fixed $T > t$. At time t, Y_t depends on the future value W_T which is not determined by $\mathcal{F}_t$ (it is independent of $\mathcal{F}_t$). Hence Y_t is not $\mathcal{F}_t$-measurable, and (Y_t) is not adapted to the Brownian filtration. In general, any process that "looks into the future" (depends on W_s for $s > t$) is not adapted.

12 Exercise 12: Linearity and Zero Mean of the Itô Integral

Let $u(t)$ and $v(t)$ be adapted processes on $[0, T]$ and c be a constant. By the linearity of the Riemann sums defining the Itô integral, one readily shows

$$\int_0^T (c\,u(t) + v(t))\,dW_t = c\int_0^T u(t)\,dW_t + \int_0^T v(t)\,dW_t.$$

Thus the Itô integral is a linear operator in the integrand. Taking expectations and using independence of increments shows immediately that

$$E\left[\int_0^T u(t)\,dW_t\right] = 0,$$

for any adapted u. Indeed, for an elementary integrand $u(t) = u_k$ on each interval, the expectation $E[u_k(W_{t_k} - W_{t_{k-1}})] = u_k E[W_{t_k} - W_{t_{k-1}}] = 0$. Passing to limits gives $E\left[\int_0^T u\,dW\right] = 0$. Hence the Itô integral has zero mean.

13 Exercise 13: Geometric Brownian Motion

The stochastic differential equation for geometric Brownian motion is

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t, \qquad S_0 \text{ given.}$$

To solve it, apply Itô's formula to $X_t = \ln S_t$. Note that

$$d(\ln S_t) = \frac{1}{S_t}\,dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2.$$

Since $dS_t = \mu S_t dt + \sigma S_t dW_t$, we have $(dS_t)^2 = \sigma^2 S_t^2 dt$. Substituting,

$$d(\ln S_t) = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}(\sigma^2 S_t^2 dt) = (\mu - \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dW_t.$$

Integrate from 0 to t to obtain

$$\ln S_t = \ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t.$$

Exponentiating yields the explicit solution:

$$S_t = S_0 \exp\left((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right).$$

Thus S_t is log-normally distributed with this mean and volatility structure.

14 Exercise 14: Itô Formula

Let X_t satisfy the SDE

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t,$$

and let $F(t,x)$ be a $C^{1,2}$-function (C^1 in t and C^2 in x). Itô's formula states that

$$dF(t, X_t) = F_t(t, X_t)\,dt + F_x(t, X_t)\,dX_t + \tfrac{1}{2}F_{xx}(t, X_t)\,(dX_t)^2.$$

Substitute $dX_t = f\,dt + g\,dW_t$. Recall that $(dW_t)^2 = dt$ in Itô calculus and $(dt)^2 = dt\,dW_t = 0$. Hence

$$(dX_t)^2 = f(t,X_t)^2(dt)^2 + 2fg\,dt\,dW_t + g(t,X_t)^2(dW_t)^2 = g(t,X_t)^2\,dt.$$

Plugging in gives the celebrated Itô formula:

$$dF(t,X_t) = \left(F_t + f\,F_x + \tfrac{1}{2}g^2\,F_{xx}\right)(t,X_t)\,dt + g(t,X_t)\,F_x(t,X_t)\,dW_t.$$

In expanded form:

$$\begin{aligned} dF(t,X_t) = {} & \frac{\partial F}{\partial t}(t,X_t)\,dt + \frac{\partial F}{\partial x}(t,X_t)f(t,X_t)\,dt \\ & + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,X_t)g(t,X_t)^2\,dt + \frac{\partial F}{\partial x}(t,X_t)g(t,X_t)\,dW_t. \end{aligned}$$

The crucial *second-order term* $\frac{1}{2}g^2F_{xx}\,dt$ has no analog in ordinary calculus. It arises because $(dW_t)^2 = dt$ rather than 0. In applications, this term (often called the Itô correction) captures the effect of Brownian motion's quadratic variation on the evolution of $F(t,X_t)$.

15 Exercise 15: Itô Formula Example ($F(x) = x^4$)

Let $X_t = W_t$ and $F(x) = x^4$. Apply Itô's formula with $F_x = 4x^3$, $F_{xx} = 12x^2$, and no explicit time dependence:

$$d(W_t^4) = 4W_t^3\,dW_t + \frac{1}{2}\cdot 12W_t^2\,(dW_t)^2 = 4W_t^3\,dW_t + 6W_t^2\,dt.$$

Equivalently,

$$W_t^4 = W_0^4 + \int_0^t 6W_s^2\,ds + \int_0^t 4W_s^3\,dW_s.$$

Since $W_0 = 0$, the first term vanishes. Taking expectations and using $E[\int_0^t W_s^3\,dW_s] = 0$ (by the martingale property of the Itô integral), we get

$$E[W_t^4] = E\left[\int_0^t 6W_s^2\,ds\right] = 6\int_0^t E[W_s^2]\,ds.$$

But $E[W_s^2] = s$ (since $W_s \sim N(0,s)$), so

$$E[W_t^4] = 6\int_0^t s\,ds = 3t^2.$$

This matches the known moment of the Gaussian distribution: in fact one can verify from the general formula $E[W_t^{2n}] = \frac{(2n)!}{n!2^n}\,t^n$ that for $n = 2$, $E[W_t^4] = 3t^2$, confirming our calculation.