Lecture 2: European Options and the Discrete Model

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Lecture Overview

This lecture covers the essential foundations for pricing European options, including:

- The nature of financial assets and derivatives in mathematical finance
- Payoff functions and the challenge of fair option pricing
- The critical market assumptions for mathematical models
- Arbitrage relationships: put-call parity and bounds
- The binomial (discrete) model and risk-neutral valuation

1. European options

Financial markets trade investments into stocks of a company, commodities (e.g. oil, gold), etc.
Stocks and commodities are risky assets, because their future value cannot be predicted. Bonds are considered as riskless assets in this lecture. If $B(t_0)$ is invested at time t_0 into a bond with a risk-free interest rate $r > 0$, then the value of the bond at time $t \geq t_0$ is simply

$$B(t) = e^{r(t-t_0)}B(t_0). \tag{1}$$

Simplifying assumption: continuous payment of interest

Spot contract: buy or sell an asset (e.g. a stock, a commodity etc.) with immediate delivery

Financial derivatives: contracts about future payments or deliveries with certain conditions

1. **forwards:** agreement between two parties to buy or sell an asset at a certain time in the future for a certain delivery price
2. **futures:** similar to forwards
3. **swaps:** contracts regulating an exchange of cash flows at different future times (e.g. currency swap, interest rate swaps, credit default swaps)
4. **options**

Example: European Call Option

At time $t = 0$ Mr. J. buys 5 European call options. Each of these options gives him the right to buy 10 shares of the company KIT at maturity $T > 0$ at the exercise price of $K = 120$ € per share.

- **Case 1:** At time $t = T$, the market price of KIT is 150 € per share.
  Mr. J. exercises his options, i.e. he buys $5 \cdot 10 = 50$ KIT shares at the price of $K = 120$ € per share and sells the shares on the market for 150 € per share.
  Hence, he wins $50 \cdot 30 = 1500$ €.
- **Case 2:** At time $t = T$, the market price of KIT is 100 € per share.
  Hence, Mr. J. does not exercise his options.

What are options good for?

- Speculation
- Hedging ("insurance" against changing market values)

Since an option gives an advantage to the holder, the option has a certain value.

For given T and K, the value $V(t, S)$ of the option must depend on the time t and the current price S of the underlying.
For a European option we know that the value at the maturity T is

$$V(T,S) = \begin{cases} (S-K)^+ := \max\{S-K, 0\} & \text{(European call)} \\ (K-S)^+ := \max\{K-S, 0\} & \text{(European put)} \end{cases}$$

The functions $S \mapsto (S-K)^+$ and $S \mapsto (K-S)^+$ are called the **payoff functions** of a call or put, respectively.

What is V?

$V(t, S(t))$ denotes the **value of the option** at time t when the stock price is $S(t)$.

- It represents the **fair price** of the option *at time* t given that the underlying stock has price $S(t)$ at that moment.
- For a European option, $V(t, S(t))$ is determined by the current time, the current stock price, and the terms of the option (such as maturity and strike price).
- Mathematically, $V(t, S(t))$ is a function that gives the option's price as the market evolves over time.
- At maturity $t = T$, $V(T, S(T))$ is simply equal to the payoff of the option, for example, $V(T, S(T)) = \max(S(T) - K, 0)$ for a call option.

In summary: $V(t, S(t))$ **is the value of the option as a function of time and the current stock price.**

The **goal of this course** is to answer the following question:

What is the fair price $V(t, S)$ of an option for $t < T$?

Why is this question important? In order to sell/buy an option, we need to know the fair price.
Why is this question non-trivial? Because the value of the risky asset is random. In particular, the price $S(T)$ at the future expiration time T is not yet known when we buy/sell the option at time $t = 0$.

1.3 Arbitrage and modelling assumptions

Example

Consider

- a stock with price $S(t)$
- a European call option with maturity $T = 1$, strike $K = 100$, and value $V(t, S(t))$
- a bond with price $B(t)$

Initial data: $S(0) = 100$, $B(0) = 100$, $V(0) = 10$.
Assumption: At time $t = 1$, we either have

$$\begin{array}{ll} \text{“up”:} & B(1) = 110,\ S(1) = 120 \\ \text{or “down”:} & B(1) = 110,\ S(1) = 80 \end{array}$$

At $t = 0$, Mrs. C. buys 0.4 bonds, one call option and sells 0.5 stock (“short selling”).
Value of the portfolio at $t = 0$:

$$0.4 \cdot B(0) + 1 \cdot V(0) - 0.5 \cdot S(0) = 0.4 \cdot 100 + 1 \cdot 10 - 0.5 \cdot 100 = 0$$

Value of the portfolio at $t = 1$ is

$$0.4 \cdot B(1) + 1 \cdot \underbrace{V(1, S(1))}_{=(S(1)-K)^+} - 0.5 \cdot S(1)$$

Two cases:

$$\begin{array}{ll} \text{“up”:} & 0.4 \cdot 110 + 1 \cdot (120 - 100)^+ - 0.5 \cdot 120 = 44 + 20 - 60 = 4 \\ \text{“down”:} & 0.4 \cdot 110 + 1 \cdot (80 - 100)^+ - 0.5 \cdot 80 = 44 + 0 - 40 = 4 \end{array}$$

In both cases, Mrs. C. wins 4 € without any risk or investment!
Why is this possible? Because the price $V(0) = 10$ of the option is too low!

Value of the Portfolio at Different Times

Value of the portfolio at $t = 0$:

- This is the *initial value* of the portfolio, i.e., the total cost (or gain) required to construct the chosen combination of financial instruments (such as stocks, bonds, and options) at the very beginning.
- If the value is zero, it means no initial investment is needed to set up the portfolio.
- If positive, you must pay this amount to create the portfolio; if negative, you receive money upon setup.

Value of the portfolio at $t = T$:

- This is the *final value* of the portfolio after all market developments (such as changes in asset prices or option maturities) have occurred.
- It represents how much wealth you have at the end, after all the contracts in the portfolio have matured and settled.
- In arbitrage examples, if this value is always positive (regardless of the market scenario) and the initial value was zero, you have achieved a risk-free profit (arbitrage).

Summary:

- The value at $t = 0$ shows the capital needed (if any) to start the portfolio.
- The value at $t = T$ shows the outcome after all possible market scenarios have played out.

How can Mrs. C. build this portfolio without investing money?

Mrs. C. is able to buy bonds and a call option, and sell stock, **without investing any money** because of the following:

- When she **sells short** 0.5 stocks at $t = 0$, she receives cash equal to $0.5 \cdot S(0)$.
- She uses this cash to **buy** 0.4 bonds and 1 call option.
- In this specific example, the cost of buying the bonds and option **exactly equals** the proceeds from short selling the stock:

$$0.4 \cdot B(0) + 1 \cdot V(0) - 0.5 \cdot S(0) = 0$$

- Thus, no initial investment is needed to set up the portfolio.

This mechanism is possible because **short selling** allows you to raise cash by selling borrowed assets, which can then be used to finance other purchases in the portfolio.

Definition 1.3.1 (Arbitrage)

Arbitrage *is the existence of a portfolio, which*

- *requires no initial investment, and*
- *which cannot cause any loss, but very likely a gain at maturity.*

Remark. A bond will always yield a risk-less gain, but it requires an investment.

Assumptions for modelling an idealized market:

(A1) Arbitrage is impossible (no-arbitrage principle)

(A2) There is a risk-free interest rate $r > 0$ which applies for all credits. Continuous payment of interest according to (1.1).

(A3) No transaction costs, taxes, etc. Trading is possible at any time. Any fraction of an asset can be sold. Liquid market, i.e. selling an asset does not change its value significantly.

(A4) A seller can sell assets he/she does not own yet ("short selling", cf. Mrs. C. above)

(A5) No dividends on the underlying asset are paid.

Remark. Discrete payment of interest: obtain $r \cdot \Delta t \cdot B(0)$ after time Δt. Value at $t = n\Delta t$:

$$\tilde{B}(t) = (1 + r \cdot \Delta t)^n B(0) = (1 + rt/n)^n B(0)$$

For $n \longrightarrow \infty$ and $\Delta t \longrightarrow 0$:

$$\lim_{n\to\infty} \tilde{B}(t) = \lim_{n\to\infty} (1 + rt/n)^n B(0) = e^{rt} B(0) = B(t)$$

(continuous payment of interest)

1.4 Arbitrage bounds

Consider European options with strike $K > 0$ and maturity T on an underlying with price $S(t)$. Let $V_P(t, S)$ and $V_C(t, S)$ be the values of a put option and call option, respectively.

Values of put and call options

Definitions:

- $S(t)$: The price of the stock at time t.
- $V_P(t, S(t))$: The value of a put option at time t, given the stock price is $S(t)$.
- $V_C(t, S(t))$: The value of a call option at time t, given the stock price is $S(t)$.

Example: Suppose at time $t = 0$:

- The stock price is $S(0) = \$100$.
- Both options have strike price $K = \$105$ and expire in one month.
- The call option value is $V_C(0, 100) = \$2$.
- The put option value is $V_P(0, 100) = \$6$.

Symbol	Meaning	Example Value
$S(0)$	Stock price now	\$100
$V_C(0, 100)$	Price of call option (strike \$105)	\$2
$V_P(0, 100)$	Price of put option (strike \$105)	\$6

Conclusion: $S(t)$ is the actual stock price. V_P and V_C are the fair market prices to buy a put or call option, respectively, at time t for the given $S(t)$.

Lemma 1 (Put-call parity)**.** *Under the assumptions (A1)-(A5) we have*

$$S(t) + V_P(t, S(t)) - V_C(t, S(t)) = e^{-r(T-t)}K$$

for all $t \in [0, T]$.

Proof. Buy one stock, buy a put, write (sell) a call. Then, the value of this portfolio is

$$\phi(t) = S(t) + V_P(t, S(t)) - V_C(t, S(t))$$

and at maturity

$$\phi(T) = S(T) + V_P(T, S(T)) - V_C(T, S(T)) = S(T) + (K - S(T))^+ - (S(T) - K)^+ = K.$$

Hence, the portfolio is risk-less. No arbitrage: The profit of the portfolio must be the same as the profit for investing $\phi(t)$ into a bond at time t:

$$\phi(T) = K \overset{!}{=} e^{r(T-t)}\phi(t) \implies e^{-r(T-t)}K = \phi(t) = S(t) + V_P(t, S(t)) - V_C(t, S(t)).$$

□

Why add V_P and subtract V_C?

In the portfolio

$$\phi(t) = S(t) + V_P(t, S(t)) - V_C(t, S(t))$$

- **"+" in front of V_P**: You *own* (are long) a put option, so you benefit if the stock price falls.
- **"−" in front of V_C**: You *sell* (are short) a call option, so you owe the call's payoff if the price rises, but you receive the call premium now.
- This combination of being long one share, long one put, and short one call "locks in" the strike price K at maturity, regardless of the stock price—just like a risk-free bond.

Why is the discounted payoff equal to the portfolio value?

The equation

$$\phi(T) = K \overset{!}{=} e^{r(T-t)}\phi(t) \implies e^{-r(T-t)}K = \phi(t) = S(t) + V_P(t, S(t)) - V_C(t, S(t))$$

means the following:

- At time T, the portfolio should deliver exactly K (for example, to replicate the payoff of a bond, or to meet a known liability).
- By investing $\phi(t)$ into the risk-free bond at time t, this will grow to $e^{r(T-t)}\phi(t)$ at time T due to compounding at the risk-free rate r.
- Therefore, to guarantee K at T, you need to invest $\phi(t) = e^{-r(T-t)}K$ at time t.
- In the specific hedging strategy given, the portfolio *value* at time t is constructed as $S(t) + V_P(t, S(t)) - V_C(t, S(t))$. This combination is chosen so that, no matter how the market evolves, its value at T will exactly be K, matching the bond payoff.

In summary: This equality expresses the principle of *replication* and *no-arbitrage*: If a portfolio is guaranteed to be worth K at time T, its value at time t must be the discounted value $e^{-r(T-t)}K$. Any other price would allow for arbitrage opportunities.

Lemma 2 (Bounds for European calls and puts)**.** *Under the assumptions (A1)-(A5), the following inequalities hold for all $t \in [0, T]$ and all $S = S(t) \geq 0$:*

$$\left(S - e^{-r(T-t)}K\right)^+ \leq V_C(t, S) \leq S \tag{1.2}$$

$$\left(e^{-r(T-t)}K - S\right)^+ \leq V_P(t, S) \leq e^{-r(T-t)}K \tag{1.3}$$

Proof. - It is obvious that $V_C(t,S) \geq 0$ and $V_P(t,S) \geq 0$ for all $t \in [0,T]$ and $S \geq 0$.

- Assume that $V_C(t,S(t)) > S(t)$ for some $S(t) \geq 0$. Write (sell) a call, buy the stock and put the difference $V_C(t,S(t)) - S(t) > 0$ in your pocket. At $t = T$, there are two scenarios: If $S(T) > K$: Must sell stock at the price K to the owner of the call. Gain: $K + V_C(t,S(t)) - S(t) > 0$ If $S(T) \leq K$: Gain $S(T) + V_C(t,S(t)) - S(t) > 0 \implies$ Arbitrage! Contradiction!

- Put-call parity:

$$S - e^{-r(T-t)}K = V_C(t,S) - \underbrace{V_P(t,S)}_{\geq 0} \leq V_C(t,S)$$

This proves (1.2). The proof of (1.3) is left as an exercise. □

What does it mean to write (sell) a call option?

To **write (sell) a call option** means you create and sell a call option to another investor.

- You receive the option price (premium) now.
- If the buyer exercises the option at maturity (i.e., if $S(T) > K$), you are obliged to sell the stock at the strike price K.
- If $S(T) \leq K$, the option is not exercised and you keep the premium.
- Your profit at expiry is:

$$\text{Profit} = \text{premium received} - \max(S(T) - K,\, 0)$$

Summary: You get money now, but may have to sell the stock at a fixed price later, even if its market value is higher.

1.5 A simple discrete model

Consider

- a stock with price $S(t)$
- a European option with maturity T, strike K, and value $V(t,S(t))$
- a bond with price $B(t) = e^{rt}B(0)$

Suppose that the initial data $S(0) = S_0$ and $B(0) = 1$ are known, and that (A1)-(A5) hold.

Goal: Find $V(0,S_0)$.

Simplifying assumption: At time $t = T$, there are only two scenarios "up": $S(T) = u \cdot S_0$ with probability p

"down": $S(T) = d \cdot S_0$ with probability $1-p$

Assumption: $0 < d < u$ and $p \in (0,1)$.
In both cases, we have $B(T) = e^{rT}B(0) = e^{rT}$.

Replication strategy: Construct portfolio with c_1 bonds and c_2 stocks such that

$$c_1 B(t) + c_2 S(t) \overset{!}{=} V(t, S(t))$$

For $t \in \{0, T\}$. For $t = T$, this means

$$\begin{aligned} \text{case "up":} \quad & c_1 e^{rT} + c_2 u S_0 \overset{!}{=} V(T, uS_0) =: V_u \\ \text{case "down":} \quad & c_1 e^{rT} + c_2 d S_0 \overset{!}{=} V(T, dS_0) =: V_d \end{aligned}$$

V_u and V_d are known if u and d are known. The unique solution is (check!)

$$c_1 = \frac{uV_d - dV_u}{(u-d)e^{rT}}, \quad c_2 = \frac{V_u - V_d}{(u-d)S_0}.$$

Hence, the fair price of the option is

$$V(0, S_0) = c_1 \underbrace{B(0)}_{=1} + c_2 S_0 = \frac{uV_d - dV_u}{(u-d)e^{rT}} + \frac{V_u - V_d}{(u-d)}.$$

which yields (check!)

$$V(0, S_0) = e^{-rT}\left(qV_u + (1-q)V_d\right) \quad \text{with} \quad q := \frac{e^{rT} - d}{u - d}. \tag{1.4}$$

Remark: The value of the option does *not* depend on p.

The no-arbitrage assumption (A1) implies $d \le e^{rT} \le u$. Hence, $q \in [0,1]$ can be seen as a probability. Now, define a new probability distribution $\mathbb{P}_q$ by

$$\mathbb{P}_q\left(S(T) = uS_0\right) = q, \qquad \mathbb{P}_q\left(S(T) = dS_0\right) = 1 - q$$

(q instead of p). Then, we have

$$\mathbb{P}_q\left(V(T, S(T)) = V_u\right) = q, \qquad \mathbb{P}_q\left(V(T, S(T)) = V_d\right) = 1 - q$$

and hence

$$qV_u + (1-q)V_d = \mathbb{E}_q\left(V(T, S(T))\right)$$

can be regarded as the **expectation** of the payoff $V(T, S(T))$ with respect to $\mathbb{P}_q$. In (1.4), this expectation is multiplied by a **discounting factor** e^{-rT}.

Interpretation: In order to have an amount of $B(t)$ at time t, we have to invest $B(0) = e^{-rT}B(t)$ into a bond at time $t = 0$.

The probability q has the property that

$$\mathbb{E}_q(S(T)) = quS_0 + (1-q)dS_0 = \frac{e^{rT}-d}{u-d}uS_0 + \frac{u-e^{rT}}{u-d}dS_0 = e^{rT}S_0.$$

Hence, the expected (with respect to $\mathbb{P}_q$) value of $S(T)$ is exactly the amount we obtain when we invest S_0 into a bond. Therefore, $\mathbb{P}_q$ is called the **risk-neutral probability**.

Meaning and Purpose of the Replication Equality

The equation

$$c_1B(t) + c_2S(t) \stackrel{!}{=} V(t, S(t))$$

means that we want to construct a portfolio consisting of c_1 units of the bond $B(t)$ and c_2 units of the stock $S(t)$ such that the total value of this portfolio **exactly matches** the value of the option $V(t, S(t))$ at all relevant times t.
Why do we need this equality?

- This is called a *replicating portfolio*: it "replicates" or mimics the payoff of the option in all possible scenarios.
- If such a portfolio exists, the **no-arbitrage principle** says the fair price of the option must be the cost to set up this portfolio. Otherwise, arbitrage opportunities would exist (risk-free profit).
- This method allows us to **determine the fair price** of the option using only the prices of traded assets (the stock and the bond), without needing to know investors' risk preferences or the real-world probabilities.

In summary: The equality ensures that the option can be perfectly hedged by a portfolio of stocks and bonds, and thus its price is uniquely determined by the absence of arbitrage.

Interpretation: What does this example tell us?

This example illustrates several foundational concepts in financial mathematics:

- **Replication Principle:** The fair price of a European option can be determined by constructing a portfolio of stocks and bonds that replicates the option's payoff in all scenarios. If such a replication is possible, the no-arbitrage price is the cost to build this portfolio.
- **Risk-Neutral Valuation:** The option price formula

$$V(0, S_0) = e^{-rT}\mathbb{E}_q\left[V(T, S(T))\right]$$

  shows that the fair value is the *discounted expected payoff*, where the expectation is taken under the **risk-neutral probability** q, not the real-world probability p.
- **Independence from Real-World Probability:** The option price does *not* depend on the real probability p of an up-move, but only on the risk-neutral probability q, which is determined by the absence of arbitrage and the interest rate.
- **No-Arbitrage and Fair Pricing:** The existence of $q \in [0, 1]$ (i.e., $d \leq e^{rT} \leq u$) ensures that there are no arbitrage opportunities. Fair pricing is based on the possibility of riskless replication, not on subjective beliefs about future market movements.
- **Risk-Neutral Measure:** Under the risk-neutral probability $\mathbb{P}_q$, the expected return of the stock equals the risk-free rate. This allows us to price derivatives by discounting expected payoffs under $\mathbb{P}_q$.
- **General Method:** This approach (binomial model, replication, risk-neutral valuation) forms the basis for more advanced models in financial mathematics, such as the Black-Scholes formula.

Conclusion: *The fair price of an option is its discounted expected payoff under the risk-neutral measure, not under the real-world probability. This is a central insight of modern mathematical finance.*

Moral of the story so far:

Clarification and Interpretation (for students)

- **Why use risk-neutral probability?** It allows us to price options as if all investors are indifferent to risk, simplifying computations and ensuring consistency with market prices.
- **Replication strategy:** By constructing a combination of stock and bond that matches the option's payoff in all scenarios, we eliminate risk and guarantee the price.
- **Discounting:** The future payoff is always discounted back to present value using the risk-free rate—this reflects the time value of money.
- **Non-dependence on p:** The actual, real-world probability p does not enter the fair price formula in the binomial model; only q (risk-neutral probability) matters for pricing under no-arbitrage.

Practice/Reflection

1. Prove the arbitrage bounds for European puts (see Lemma 2).
2. For $S_0 = 100$, $u = 1.2$, $d = 0.9$, $r = 0.05$, $T = 1$, $K = 100$, compute q and $V(0, S_0)$ for a European call option.
3. Why does the price in the binomial model not depend on p?
4. In your own words, explain the meaning of risk-neutral probability and why it is fundamental in financial mathematics.

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