Lecture 5: Risk-neutral Valuation and Numerical Approximation of SDE

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Today's Agenda

1. Risk-neutral valuation and its motivation.
2. The Black–Scholes PDE and discounted expectations.
3. Equivalent martingale measures.
4. The Feynman–Kac theorem and its application to Black–Scholes.
5. Interpretation and implications for option pricing.
6. Numerical methods for stochastic differential equations.

1. Risk-neutral valuation and equivalent martingale measures

In Section 1.5, we have seen that in the simplified two-scenario model, the value of an option can be priced by replication. The same strategy was applied to the refined model in the previous section. In the simple situation considered in 1.5, the value of an option turned out to be the discounted expectation of the payoff under the risk-neutral probability. In this subsection, we will see that this is also true for the refined model from Section 3.2.

Theorem 3.4.1 (Option price as discounted expectation)

If $V(t,S)$ is the solution of the Black–Scholes equation

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T],\ S > 0$$

$$V(T, S) = \psi(S)$$

with payoff function $\psi(S)$, then

$$V(t_\star, S_\star) = e^{-r(T-t_\star)} \int_0^\infty \psi(x)\phi(x; \xi, \beta)\, dx \tag{3.7}$$

for all $t_\star \in [0, T]$ and $S_\star > 0$. The function ϕ is the density of the log-normal distribution (cf. Definition 3.1.2) with parameters

$$\xi = \ln S_\star + \left(r - \frac{\sigma^2}{2}\right)(T - t_\star), \qquad \beta = \sigma\sqrt{T - t_\star}. \tag{3.8}$$

The assertion can be shown by showing that the above representation coincides with the Black–Scholes formulas for puts and calls. Such a proof, however, involves several changes of variables in the integral representations and rather tedious calculations. We give a shorter and more elegant proof:

Proof.

Step 1: In our derivation of the Black–Scholes model, we have assumed that

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

i.e., that the price of the underlying is a geometric Brownian motion with drift μS_t; cf. (3.1). It turned out, however, that the parameter μ does not appear in the Black–Scholes equation. Hence, we can choose $\mu = r$ and consider the SDE

$$d\widehat{S}_t = r\widehat{S}_t dt + \sigma \widehat{S}_t dW_t, \qquad t \in [t_\star, T]$$

with initial condition

$$\widehat{S}_{t_\star} = S_\star$$

as a model for the stock price.

Step 2: The function $u(t, S) := e^{r(T-t)}V(t, S)$ solves the PDE

$$\partial_t u(t, S) + \frac{\sigma^2}{2} S^2 \partial_S^2 u(t, S) + rS\partial_S u(t, S) = 0, \qquad t \in [0, T]$$

because

$$\begin{aligned}
&\partial_t u(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 u(t,S) + rS\partial_S u(t,S) \\
&= -re^{r(T-t)} V(t,S) + e^{r(T-t)} \partial_t V(t,S) \\
&\quad + \frac{\sigma^2}{2} S^2 e^{r(T-t)} \partial_S^2 V(t,S) + rSe^{r(T-t)} \partial_S V(t,S) \\
&= e^{r(T-t)} \left(-rV(t,S) + \partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) \right) \\
&= 0 \qquad \text{(by Black–Scholes equation)}
\end{aligned}$$

Moreover, u satisfies the terminal condition

$$u(T,S) = V(T,S) = \psi(S).$$

Step 3: Applying the Feynman–Kac formula (cf. 2.6) with $f(t,S) = rS$ and $g(t,S) = \sigma S$ yields

$$\mathbb{E}\left(\psi(\widehat{S}_T)\right) = u(t_\star, S_\star) = e^{r(T-t_\star)} V(t_\star, S_\star)$$

and thus

$$V(t_\star, S_\star) = e^{-r(T-t_\star)} \mathbb{E}\left(\psi(\widehat{S}_T)\right).$$

We know that $\widehat{S}_T$ is log-normal, i.e.,

$$\mathbb{E}\left(\psi(\widehat{S}_T)\right) = \int_0^\infty \psi(x)\phi(x;\xi,\beta)\,dx$$

with ϕ as above.

Interpretation

We know from Section 3.1 that

$$\begin{aligned}
\mathbb{E}\left(\widehat{S}_T\right) &= \int_0^\infty x\phi(x;\xi,\beta)\,dx \\
&= \exp\left(\xi + \frac{\beta^2}{2}\right) \\
&= \exp\left(\ln S_\star + \left(r - \frac{\sigma^2}{2}\right)(T - t_\star) + \frac{1}{2}\left(\sigma\sqrt{T-t_\star}\right)^2\right) \\
&= \exp\left(\ln S_\star + r(T-t_\star)\right) \\
&= S_\star \exp\left(r(T-t_\star)\right)
\end{aligned}$$

This means that for $\mu = r$ the expected value of the stock is exactly the money obtained by investing $S_\star$ into a bond at time $t_\star$ and waiting until $T - t_\star$. Hence, the log-normal distribution with parameters (3.8) defines the **risk-neutral probability**; cf. 1.5.

The integral in (3.7) is precisely the expected payoff under the risk-neutral probability, and (3.7) states that the price of the option is obtained by discounting the expected payoff.

2. Numerical methods for stochastic differential equations

2.1 Motivation

According to 3.4 the value of a European option is the discounted expected payoff under the risk-neutral probability:

$$V(0, S_0) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(\psi\left(S(T)\right)\right)$$

For the standard Black–Scholes model:

$$V(0, S_0) = e^{-rT}\int_0^{\infty}\psi(x)\phi(x,\xi,\beta)\,dx$$

with log-normal density ϕ and parameters

$$\xi = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \qquad \beta = \sigma\sqrt{T}.$$

A way to price the option:

1. **Monte-Carlo method.** In the Black–Scholes model, $S(t)$ is defined by the SDE

   $$dS(t) = rS(t)dt + \sigma S(t)dW(t), \qquad t \in [0,T], \quad S_0 \text{ given}$$

   (risk-neutral, $\mu = r$)

   Solution: Geometric Brownian motion

   $$S(t) = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right).$$

   This is the process which corresponds to $\phi(x,\xi,\beta)$, because $S(T)$ is log-normal with the same parameters. Estimate the expected payoff as follows:

   - Generate many realizations $S(T,\omega_1),\dots,S(T,\omega_m)$, $m \in \mathbb{N}$ "large".
   - Approximate

     $$V(0,S_0) \approx e^{-rT}\frac{1}{m}\sum_{j=1}^{m}\psi\left(S(T,\omega_j)\right)$$

Consider now a more complicated price process:

$$dS(t) = rS(t)dt + \sigma(t)dW^1(t) \tag{5.1a}$$

$$d\sigma^2(t) = \kappa\left(\theta - \sigma^2(t)\right)dt + \nu\left(\rho dW^1(t) + \sqrt{1-\rho^2}dW^2(t)\right) \tag{5.1b}$$

Heston model with parameters $r, \kappa, \theta, \nu > 0$, initial values S_0, σ_0, independent scalar Wiener processes $W^1(t), W^2(t)$, correlation $\rho \in [-1, 1]$
Steven L. Heston 1993

Now the volatility is not a parameter, but a stochastic process defined by a second SDE. We do not have an explicit formula for $S(t)$ and $\sigma(t)$, but the Monte-Carlo approach is still feasible:

- Choose $N \in \mathbb{N}$, define step-size $\tau = T/N$ and $t_n = n\tau$. For each $\omega_1, \ldots, \omega_m$ compute approximations

  $$X_n^1(\omega_j) \approx S(t_n, \omega_j), \qquad X_n^2(\omega_j) \approx \sigma^2(t_n, \omega_j), \qquad n = 0, \ldots, N$$

  by solving the SDEs (5.1a), (5.1b) numerically.
- Approximate
  $$V(0, S_0) \approx e^{-rT}\frac{1}{m}\sum_{j=1}^{m}\psi\left(X_N^1(\omega_j)\right)$$

The Monte-Carlo approach even works for other types of options. As an example, consider an Asian option with payoff

$$\psi\left(t \mapsto S(t)\right) = \left(S(T) - \frac{1}{T}\int_0^T S(t)\,dt\right)^+ \qquad \text{(average strike call).}$$

Now the payoff depends on the entire path $t \mapsto S(t)$. We approximate

$$S(T, \omega_j) \approx X_N^1(\omega_j), \qquad \frac{1}{T}\int_0^T S(t, \omega_j)\,dt \approx \frac{1}{N}\sum_{n=1}^{N} X_n^1(\omega_j)$$

and hence

$$V(0, S_0) \approx e^{-rT}\frac{1}{m}\sum_{j=1}^{m}\left(X_N^1(\omega_j) - \frac{1}{N}\sum_{n=1}^{N} X_n^1(\omega_j)\right)^+$$

Remark: In the original paper, Heston derives an explicit Black-Scholes-type formula for European options by means of characteristic functions. Hence, European options in the Heston model can also be priced by quadrature formulas, but for Asian options this is impossible.

Goal: Construct and analyze numerical methods for SDEs.

3. Euler-Maruyama method

3.1 Derivation

Consider the one-dimensional SDE

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \qquad t \in [0, T], \qquad X(0) = X_0$$

with suitable functions f and g and a given initial value X_0. Choose $N \in \mathbb{N}$, define step-size $\tau = T/N$ and $t_n = n\tau$.

$$\begin{aligned} X(t_{n+1}) &= X(t_n) + \int_{t_n}^{t_{n+1}} f(s, X(s))\, ds + \int_{t_n}^{t_{n+1}} g(s, X(s))\, dW(s) \\ &\approx X(t_n) + (t_{n+1} - t_n) f(t_n, X(t_n)) + g(t_n, X(t_n)) \underbrace{(W(t_{n+1}) - W(t_n))}_{=:\Delta W_n} \end{aligned}$$

Replacing $X(t_n) \longrightarrow X_n$ and "$\approx$" $\longrightarrow$ "$=$" yields the

Euler-Maruyama method (Gisiro Maruyama 1955, Leonhard Euler 1768-70): For $n = 0, \ldots, N-1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

$$X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n)\Delta W_n.$$

Hope that $X_n \approx X(t_n)$.

	SDE	recursion
$X(t)$	exact	approx.
X_n	approx.	exact

$\Longrightarrow \qquad X(t_n) \approx X_n$

The exact solution $X(t_n)$ and the numerical approximation X_n are random variables. For every path $t \mapsto W(t, \omega)$ of the Wiener process, a different result is obtained. $X(t)$ is called **strong solution** if $t \mapsto W(t, \omega)$ is given, and **weak solution** if $t \mapsto W(t, \omega)$ can be chosen. Approximations of weak solutions: For each n, generate a random number $Z_n \sim \mathcal{N}(0, 1)$ and let

$$\Delta W_n = \sqrt{\tau} Z_n.$$

Question: Does X_n really approximate $X(t_n)$? In which sense? How accurately?

3.2 Strong convergence

Definition 5.2.1 (strong convergence)
Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$ and $t_n = n\tau$. An approximation $X_n(\omega) \approx X(t_n, \omega)$ converges

- **strongly** with order $\gamma > 0$, if there is a constant $C > 0$ independent of τ such that

$$\max_{n=0,\ldots,N} \mathbb{E}\left(|X(t_n) - X_n|\right) \leq C\tau^\gamma$$

for all sufficiently small τ.

3.3 Existence and uniqueness of solutions of SDEs

Theorem 5.2.2 (existence and uniqueness)
Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be functions with the following properties:

- **Lipschitz condition:** There is a constant $L \geq 0$ such that
$$|f(t,x) - f(t,y)| \leq L|x-y|, \qquad |g(t,x) - g(t,y)| \leq L|x-y|$$
for all $x, y \in \mathbb{R}$ and $t \geq 0$.
- **Linear growth condition:** There is a constant $K \geq 0$ such that
$$|f(t,x)|^2 \leq K(1+|x|^2), \qquad |g(t,x)|^2 \leq K(1+|x|^2)$$
for all $x \in \mathbb{R}$ and $t \geq 0$.

Then, the SDE

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \qquad t \in [0,T]$$

with deterministic initial value $X(0) = X_0$ has a continuous adapted solution and

$$\sup_{t \in [0,T]} \mathbb{E}\left(X^2(t)\right) < \infty.$$

If both $X(t)$ and $\widetilde{X}(t)$ are such solutions, then

$$\mathbb{P}\left(X(t) = \widetilde{X}(t) \text{ for all } t \in [0,T]\right) = 1.$$

Proof: exercise.
Remark: The assumptions can be weakened.

3.4 Strong convergence of the Euler-Maruyama method

For simplicity, we only consider the autonomous SDE

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \qquad t \in [0,T]$$

and the Euler–Maruyama approximation

$$X_{n+1} = X_n + \tau f(X_n) + g(X_n)\Delta W_n.$$

with $X(0) = X_0$, $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $t_n = n\tau$.

We assume that $f = f(x)$ and $g = g(x)$ satisfy the Lipschitz condition (5.2). In the autonomous case, this implies the linear growth condition (5.3) (exercise).

Theorem 5.2.3 (strong error of the Euler-Maruyama method) *Under these conditions, there is a constant $\hat{C}$ such that*

$$\max_{n=0,\dots,N} \mathbb{E}\left(|X(t_n) - X_n|\right) \leq \hat{C}\tau^{1/2}$$

for all sufficiently small τ. $\hat{C}$ does not depend on τ.

For the proof see the following.

Lemma 5.2.4 (Gronwall)

Let $\alpha : [0, T] \to \mathbb{R}_+$ be a positive integrable function. If there are constants $a > 0$ and $b > 0$ such that

$$0 \leq \alpha(t) \leq a + b \int_0^t \alpha(s)\, ds$$

for all $t \in [0, T]$, then $\alpha(t) \leq ae^{bt}$.

Proof: exercise.

Proof of Theorem 5.2.3.

Strategy:

- Define the step function

  $$Y(t) = \sum_{n=0}^{N-1} \mathbf{1}_{[t_n,t_{n+1})}(t) X_n \qquad \text{for } t \in [0, T), \qquad Y(T) := X_N.$$

  For $n = 0, \dots, N-1$ this means that

  $$Y(t) = X_n \iff t \in [t_n, t_{n+1}).$$

- Define $\alpha(s) := \sup_{r\in[0,s]} \mathbb{E}(|Y(r) - X(r)|^2)$ and prove the Gronwall inequality

  $$0 \leq \alpha(t) \leq C\tau + b \int_0^t \alpha(s)\, ds. \tag{5.4}$$

- Apply Gronwall's lemma. This yields $\alpha(t) \leq \tau\hat{C}^2$ with $\hat{C}^2 = Ce^{bt}$.

- Since[1] $\mathbb{E}(Z) \leq \sqrt{\mathbb{E}(Z^2)}$ for random variables Z, it follows that

  $$\begin{aligned}\max_{n=0,\dots,N} \mathbb{E}\left(|X_n - X(t_n)|\right) &\leq \sup_{t\in[0,T]} \mathbb{E}\left(|Y(t) - X(t)|\right) \\ &\leq \sup_{t\in[0,T]} \sqrt{\mathbb{E}\left(|Y(t) - X(t)|^2\right)} = \sqrt{\alpha(T)} \leq \sqrt{\tau}\hat{C}\end{aligned}$$

Main challenge: Prove Gronwall inequality (5.4). Choose fixed $t \in [0,T]$ and let n be the index with $t \in [t_n, t_{n+1})$.

Derive integral representation of the error:

$$\begin{aligned} Y(t) = X_n = X_0 + \sum_{k=0}^{n-1}(X_{k+1} - X_k) &= X_0 + \sum_{k=0}^{n-1}\Big(\tau f(X_k) + g(X_k)\Delta W_k\Big) \\ &= X_0 + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} f(X_k)\,ds + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g(X_k)\,dW(s) \\ &= X_0 + \int_0^{t_n} f(Y(s))\,ds + \int_0^{t_n} g(Y(s))\,dW(s) \end{aligned}$$

Comparing with the exact solution

$$X(t) = X(0) + \int_0^t f(X(s))\,ds + \int_0^t g(X(s))\,dW(s)$$

yields the error representation

$$\begin{aligned} Y(t) - X(t) &= \underbrace{\int_0^{t_n} [f(Y(s)) - f(X(s))]\,ds}_{=:\mathcal{T}_1} + \underbrace{\int_0^{t_n} [g(Y(s)) - g(X(s))]\,dW(s)}_{=:\mathcal{T}_2} \\ &\quad - \underbrace{\int_{t_n}^t f(X(s))ds}_{=:\mathcal{T}_3} - \underbrace{\int_{t_n}^t g(X(s))dW(s)}_{=:\mathcal{T}_4} \\ &= \mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - \mathcal{T}_4. \end{aligned}$$

The Cauchy-Schwarz inequality gives

$$(\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - \mathcal{T}_4)^2 = ((1,1,-1,-1)T)^2 \leq 4\|T\|_2^2 = 4\cdot(\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2 + \mathcal{T}_4^2)$$

and hence

$$\mathbb{E}|Y(t) - X(t)|^2 \leq 4\cdot\mathbb{E}\left(\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2 + \mathcal{T}_4^2\right).$$

First term: For functions $u \in L^2([0,t_n])$ the Cauchy-Schwarz inequality yields

$$\left(\int_0^{t_n} u(s)\cdot 1\,ds\right)^2 \leq \int_0^{t_n} |u(s)|^2\,ds \cdot \int_0^{t_n} 1^2\,ds = \int_0^{t_n} |u(s)|^2\,ds \cdot t_n. \tag{5.5}$$

[01]Elementary calculation: $0 \leq \mathbb{V}(Z) = \mathbb{E}\left([Z - \mathbb{E}(Z)]^2\right) = \mathbb{E}\left[Z^2 - 2Z\mathbb{E}(Z) + (\mathbb{E}(Z))^2\right] = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2$ and hence $(\mathbb{E}(Z))^2 \leq \mathbb{E}(Z^2)$.

Using the Lipschitz bound (5.2), we obtain

$$\begin{aligned}
\mathbb{E}(\mathcal{T}_1^2) &= \mathbb{E}\left[\left(\int_0^{t_n} [f(Y(s)) - f(X(s))]\, ds\right)^2\right] \\
&\le t_n \mathbb{E}\left(\int_0^{t_n} |f(Y(s)) - f(X(s))|^2\, ds\right) \\
&\le TL^2 \int_0^{t_n} \mathbb{E}(|Y(s) - X(s)|^2)\, ds \\
&\le TL^2 \int_0^t \alpha(s)\, ds \qquad (\text{since } t \ge t_n).
\end{aligned}$$

Second term: It follows from the Itô isometry (Theorem 2.3.5) and the Lipschitz bound (5.2) that

$$\begin{aligned}
\mathbb{E}(\mathcal{T}_2^2) &= \mathbb{E}\left(\int_0^{t_n} [g(Y(s)) - g(X(s))]\, dW(s)\right)^2 \\
&= \mathbb{E}\left(\int_0^{t_n} |g(Y(s)) - g(X(s))|^2\, ds\right) \\
&\le L^2 \int_0^{t_n} \mathbb{E}(|Y(s) - X(s)|^2)\, ds \\
&\le L^2 \int_0^t \alpha(s)\, ds \qquad (\text{since } t \ge t_n).
\end{aligned}$$

Third term: Equation (5.5) and the linear growth bound (5.3) yield

$$\begin{aligned}
\mathbb{E}(\mathcal{T}_3^2) &= \mathbb{E}\left[\left(\int_{t_n}^t f(X(s))\, ds\right)^2\right] \\
&\le (t - t_n)\mathbb{E}\left(\int_{t_n}^t |f(X(s))|^2\, ds\right) \\
&\le \tau K \cdot \mathbb{E}\left(\int_{t_n}^t (1 + |X(s)|^2)\, ds\right) \le c\tau^2
\end{aligned}$$

because Theorem 5.2.2 states that $\mathbb{E}(1 + |X(s)|^2)$ remains bounded on $[t_n, t]$.

Last term: Using the Itô isometry and the linear growth bound (5.3) it follows that

$$\begin{aligned}
\mathbb{E}(\mathcal{T}_4^2) &= \mathbb{E}\left[\left(\int_{t_n}^t g(X(s))\, dW(s)\right)^2\right] \\
&\le \mathbb{E}\left(\int_{t_n}^t |g(X(s))|^2\, ds\right) \\
&\le K \cdot \mathbb{E}\left(\int_{t_n}^t (1 + |X(s)|^2)\, ds\right) \le c\tau
\end{aligned}$$

These bounds yield the Gronwall inequality (5.4) with $b = 4(T+1)L^2$ and with C depending on K and $\sup_{s\in[0,T]} \mathbb{E}(1 + |X(s)|^2)$.