Lecture 5 (Remaining part): Existence and Uniqueness of Solutions for SDEs: Full Detailed Proof with All Assumptions

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Consider the stochastic differential equation (SDE)

$$dX(t) = f(t, X(t))\, dt + g(t, X(t))\, dW(t), \qquad t \in [0, T], \qquad X(0) = X_0,$$

where:

- $X(t) \in \mathbb{R}^d$ for all $t \in [0, T]$,
- $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function (the drift),
- $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ is a measurable function (the diffusion),
- $W = (W_1, \dots, W_m)$ is a standard m-dimensional Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$,
- X_0 is an $\mathcal{F}_0$-measurable random variable in $L^2(\Omega; \mathbb{R}^d)$, i.e., $\mathbb{E}[\|X_0\|^2] < \infty$.

We assume the following conditions on the coefficients f and g:

- **(Lipschitz condition)** There exists a constant $L > 0$ such that for all $t \in [0, T]$ and all $x, y \in \mathbb{R}^d$,

$$\|f(t, x) - f(t, y)\| + \|g(t, x) - g(t, y)\| \leq L\|x - y\|.$$

- **(Linear growth condition)** There exists a constant $K > 0$ such that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$\|f(t, x)\|^2 + \|g(t, x)\|^2 \leq K(1 + \|x\|^2).$$

- Both f and g are measurable and adapted: for each t, $f(t, \cdot)$ and $g(t, \cdot)$ are Borel measurable in x, and for each x, $f(\cdot, x)$ and $g(\cdot, x)$ are measurable in t.

Theorem 1 *Under these conditions, there exists a unique adapted continuous process $X(t)$, $t \in [0, T]$, which is $\mathbb{R}^d$-valued, such that*

$$X(t) = X_0 + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s) \qquad \forall t \in [0, T],$$

and

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|X(t)\|^2\right] < \infty.$$

Proof: *Step 1: Construction via Picard iteration.* Define a sequence of processes $\{X^{(k)}(t)\}_{k=0}^{\infty}$ recursively. Set $X^{(0)}(t) := X_0$ for all $t \in [0,T]$. For $k \geq 0$, define

$$X^{(k+1)}(t) := X_0 + \int_0^t f(s, X^{(k)}(s))\, ds + \int_0^t g(s, X^{(k)}(s))\, dW(s),$$

where the stochastic integral is an Itô integral. *Step 2: Uniform moment bounds for each Picard iterate.* We prove by induction that for each k,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X^{(k)}(t)\|^2\right] < \infty.$$

For $k = 0$, since $X^{(0)}(t) = X_0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X^{(0)}(t)\|^2\right] = \mathbb{E}[\|X_0\|^2] < \infty.$$

Assume the bound holds for some $k \geq 0$. For $X^{(k+1)}(t)$, apply the triangle inequality:

$$\|X^{(k+1)}(t)\|^2 \leq 3\left(\|X_0\|^2 + \left\|\int_0^t f(s, X^{(k)}(s))\, ds\right\|^2 + \left\|\int_0^t g(s, X^{(k)}(s))\, dW(s)\right\|^2\right).$$

Taking the supremum and expectation,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \|X^{(k+1)}(t)\|^2\right] &\leq 3\mathbb{E}[\|X_0\|^2] + 3\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t f(s, X^{(k)}(s))\, ds\right\|^2\right] \\ &\quad + 3\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t g(s, X^{(k)}(s))\, dW(s)\right\|^2\right].\end{aligned}$$

Using Jensen's inequality for the deterministic integral and Doob's martingale inequality for the stochastic integral,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t f(s, X^{(k)}(s))\, ds\right\|^2\right] \leq T\mathbb{E}\left[\int_0^T \|f(s, X^{(k)}(s))\|^2\, ds\right],$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t g(s, X^{(k)}(s))\, dW(s)\right\|^2\right] \leq 4\mathbb{E}\left[\int_0^T \|g(s, X^{(k)}(s))\|^2\, ds\right].$$

By the linear growth condition,

$$\|f(s, X^{(k)}(s))\|^2 + \|g(s, X^{(k)}(s))\|^2 \leq K(1 + \|X^{(k)}(s)\|^2),$$

so

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X^{(k+1)}(t)\|^2\right] \leq 3\mathbb{E}[\|X_0\|^2] + 3TK\int_0^T (1 + \mathbb{E}[\|X^{(k)}(s)\|^2])\, ds + 12K\int_0^T (1 + \mathbb{E}[\|X^{(k)}(s)\|^2])\, ds.$$

Since $\mathbb{E}[\|X^{(k)}(s)\|^2] \leq \mathbb{E}\left[\sup_{r\in[0,T]} \|X^{(k)}(r)\|^2\right]$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X^{(k+1)}(t)\|^2\right] \leq C_1 + C_2\int_0^T \mathbb{E}\left[\sup_{r\in[0,s]} \|X^{(k)}(r)\|^2\right] ds,$$

where $C_1 = 3\mathbb{E}[\|X_0\|^2]+(3TK+12K)T$ and $C_2 = 3TK+12K$. By Gronwall's lemma, since the induction hypothesis ensures the integral is finite,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X^{(k+1)}(t)\|^2\right] \le C,$$

for some constant C depending on $T, K, \mathbb{E}[\|X_0\|^2]$. *Step 3: Cauchy property of the Picard sequence.* Define $Y^{(k)}(t) := X^{(k+1)}(t) - X^{(k)}(t)$. Then,

$$Y^{(k)}(t) = \int_0^t [f(s, X^{(k)}(s)) - f(s, X^{(k-1)}(s))]\, ds + \int_0^t [g(s, X^{(k)}(s)) - g(s, X^{(k-1)}(s))]\, dW(s).$$

Using the triangle inequality,

$$\|Y^{(k)}(t)\|^2 \le 2\left(\left\|\int_0^t [f(s, X^{(k)}(s)) - f(s, X^{(k-1)}(s))]\, ds\right\|^2 + \left\|\int_0^t [g(s, X^{(k)}(s)) - g(s, X^{(k-1)}(s))]\, dW(s)\right\|^2\right).$$

Taking the supremum and expectation, and applying Jensen's and the Burkholder-Davis-Gundy (BDG) inequalities,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \|Y^{(k)}(t)\|^2\right] \le 2T\mathbb{E}&\left[\int_0^T \|f(s, X^{(k)}(s)) - f(s, X^{(k-1)}(s))\|^2\, ds\right] \\ &+ 8\mathbb{E}\left[\int_0^T \|g(s, X^{(k)}(s)) - g(s, X^{(k-1)}(s))\|^2\, ds\right].\end{aligned}$$

By the Lipschitz condition,

$$\|f(s, X^{(k)}(s)) - f(s, X^{(k-1)}(s))\|^2 + \|g(s, X^{(k)}(s)) - g(s, X^{(k-1)}(s))\|^2 \le 2L^2\|Y^{(k-1)}(s)\|^2,$$

so

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Y^{(k)}(t)\|^2\right] \le (2TL^2 + 8L^2)\int_0^T \mathbb{E}\left[\|Y^{(k-1)}(s)\|^2\right]\, ds = C_3\int_0^T \mathbb{E}\left[\|Y^{(k-1)}(s)\|^2\right]\, ds,$$

where $C_3 = 2TL^2 + 8L^2$. Let $a_k := \sup_{t\in[0,T]} \mathbb{E}[\|Y^{(k)}(t)\|^2]$. Then,

$$a_k \le C_3 T a_{k-1}.$$

For $C_3T < 1$ (or by splitting $[0, T]$ into smaller intervals), $a_k \to 0$ as $k \to \infty$. Thus, $\{X^{(k)}\}$ is a Cauchy sequence in $L^2(\Omega; C([0, T]; \mathbb{R}^d))$ and converges to a limit X. *Step 4: X is a solution.* Since the integral operators are continuous in L^2, taking the limit as $k \to \infty$,

$$X(t) = X_0 + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s).$$

Step 5: Uniqueness. Assume two solutions $X(t)$ and $\tilde{X}(t)$ with $X(0) = \tilde{X}(0) = X_0$. Define $Z(t) = X(t) - \tilde{X}(t)$. Then,

$$Z(t) = \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))]\, ds + \int_0^t [g(s, X(s)) - g(s, \tilde{X}(s))]\, dW(s),$$

with $Z(0) = 0$. Define

$$u(t) = \mathbb{E}\left[\sup_{s\in[0,t]} \|Z(s)\|^2\right].$$

We aim to show $u(t) = 0$ for all $t \in [0,T]$.First, note that

$$\sup_{s\in[0,t]} \|Z(s)\| \leq \sup_{s\in[0,t]} \left\|\int_0^s [f(r,X(r)) - f(r,\tilde{X}(r))]\,dr\right\| + \sup_{s\in[0,t]} \left\|\int_0^s [g(r,X(r)) - g(r,\tilde{X}(r))]\,dW(r)\right\|.$$

Using $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} \|Z(s)\|^2\right] \leq 2\mathbb{E}&\left[\sup_{s\in[0,t]} \left\|\int_0^s [f(r,X(r)) - f(r,\tilde{X}(r))]\,dr\right\|^2\right] \\ &+ 2\mathbb{E}\left[\sup_{s\in[0,t]} \left\|\int_0^s [g(r,X(r)) - g(r,\tilde{X}(r))]\,dW(r)\right\|^2\right].\end{aligned}$$

Drift Term:

$$\left\|\int_0^s [f(r,X(r)) - f(r,\tilde{X}(r))]\,dr\right\| \leq \int_0^s \|f(r,X(r)) - f(r,\tilde{X}(r))\|\,dr \leq L\int_0^s \|Z(r)\|\,dr.$$

Then,

$$\left\|\int_0^s [f(r,X(r)) - f(r,\tilde{X}(r))]\,dr\right\|^2 \leq \left(L\int_0^s \|Z(r)\|\,dr\right)^2 \leq L^2 s\int_0^s \|Z(r)\|^2\,dr \leq L^2 t\int_0^s \|Z(r)\|^2\,dr.$$

Thus,

$$\mathbb{E}\left[\sup_{s\in[0,t]} \left\|\int_0^s [f(r,X(r)) - f(r,\tilde{X}(r))]\,dr\right\|^2\right] \leq L^2 t\int_0^t \mathbb{E}\left[\|Z(r)\|^2\right]\,dr.$$

Diffusion Term: By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} \left\|\int_0^s [g(r,X(r)) - g(r,\tilde{X}(r))]\,dW(r)\right\|^2\right] &\leq 4\mathbb{E}\left[\int_0^t \|g(r,X(r)) - g(r,\tilde{X}(r))\|^2\,dr\right] \\ &\leq 4L^2\int_0^t \mathbb{E}\left[\|Z(r)\|^2\right]\,dr.\end{aligned}$$

Combining both terms,

$$u(t) \leq 2L^2 t\int_0^t \mathbb{E}\left[\|Z(r)\|^2\right]\,dr + 8L^2\int_0^t \mathbb{E}\left[\|Z(r)\|^2\right]\,dr \leq (2L^2T + 8L^2)\int_0^t u(r)\,dr,$$

since $\mathbb{E}\left[\|Z(r)\|^2\right] \leq u(r)$ and $t \leq T$. Let $K = 2L^2T + 8L^2$. Then,

$$u(t) \leq K\int_0^t u(r)\,dr.$$

Since $u(0) = 0$, by Gronwall's lemma,

$$u(t) \leq 0 \cdot e^{Kt} = 0.$$

Thus, $u(t) = 0$ for all $t \in [0,T]$, implying $\sup_{t\in[0,T]} \|Z(t)\|^2 = 0$ almost surely. Therefore, $Z(t) = 0$ almost surely for all t, proving uniqueness.

Step 6: Moment bound. From Step 2, the uniform bound on $\{X^{(k)}\}$ and convergence in $L^2(\Omega; C([0,T];\mathbb{R}^d))$

imply

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|^2\right] < \infty.$$

$\square$