Lecture 1: Introduction to Mathematical Finance

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Welcome!

Welcome to Mathematical Finance!
This course will teach you how mathematics, especially probability and statistics, help us understand and operate in financial markets.
Even if you have never studied finance before, don't worry—we begin from the basics!

In this first lecture, our aim is to build an intuition for why mathematics is so valuable in finance, and how even simple mathematical tools can lead to smarter decisions and real advantages in the world of money.

Today's Goals:

- Introduce what mathematical finance is and why it matters
- Motivate with simple, practical examples
- Outline the key concepts and roadmap of the course

1. What is Mathematical Finance?

Definition

Mathematical finance is the science of using mathematical methods to solve problems in finance. It helps answer questions like:

- How do we value financial products, like stocks or options?
- How do we measure and manage risk?
- How can we make rational investment decisions when the future is uncertain?

Why do we need mathematics? *In real life, financial decisions are rarely certain. Mathematics gives us a logical, structured way to analyze situations where the outcome isn't obvious. It helps to avoid mistakes that can cost real money!*

Think about this: If you were offered two different investment opportunities, how could you decide which one is best? What numbers or information would you want to know? (Let's keep these questions in mind as we proceed.)

2. Why Use Mathematics in Finance?

- Financial markets are unpredictable—prices move randomly.
- We want to make decisions not just based on intuition, but on logical reasoning.
- Mathematics provides tools to analyze uncertainty, estimate risks, and find optimal strategies.

Real-World Example

Investing in stocks, pricing insurance, managing risk at banks—all use quantitative methods.

Remark: In fact, most big decisions in finance—from how banks lend money to how companies invest billions—are guided by mathematical models and calculations. Even the ”fair price” of a simple insurance policy relies on math!

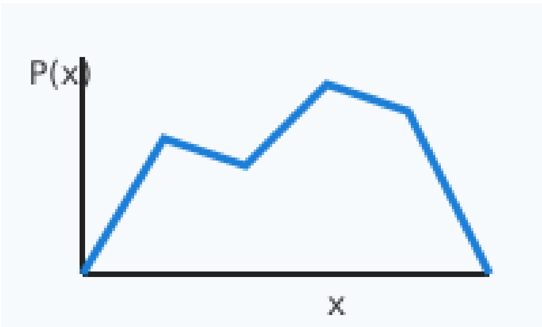
3. Role of Probability Theory

Core Tool

Probability is our main tool for modelling uncertainty in finance. We use probability to:

- Describe possible future prices of assets
- Calculate expected returns and risks
- Price and hedge financial derivatives (e.g., options)

Why probability? Because in finance, we rarely know what will happen in the future. Instead, we can list all possible outcomes and assign each a probability—like tossing a coin, rolling dice, or predicting tomorrow's stock price.



Pause for thought: *Suppose you have a 60% chance to win and a 40% chance to lose in a game—would you play? How can you know if the game is "worth it"?*

4. Motivating Example: Coin Toss Investment

Game Example

Game: Toss a biased coin. Heads (60%): Win \$10. Tails (40%): Lose \$10

Let's analyze: Should you play this game?

Calculate the *expected profit*:

$$\mathbb{E}[\text{Profit}] = 0.6 \times 10 + 0.4 \times (-10) = 2$$

Mathematical analysis reveals profitable opportunities that intuition alone might miss!

Explanation: Even though you can lose, if you play this game many times, on average you will win \$2 per play. This is the power of **expectation**: it helps you see past the uncertainty to the true value of a risky opportunity.

In finance, every investment is like a game with uncertain outcomes. We use expectation to find out if it's truly "worth it".

5. Course Roadmap: What You'll Learn

Key Concepts

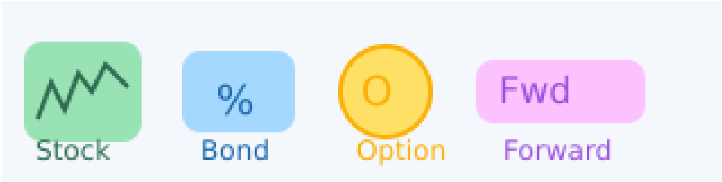
- Random Variables, Expectation, and Variance
- No-Arbitrage Principle
- Pricing of Financial Derivatives (like options)
- Hedging strategies to manage risk
- Real market models: Binomial Model, Black-Scholes, and more

We'll start with basics, build up concepts step by step, and see how everything connects through practical examples and real-world cases.

6. Fundamentals: Financial Instruments

- **Stock:** A share in the ownership of a company.
- **Bond:** A fixed-income loan made by an investor to a borrower.
- **Option:** Right (but not obligation) to buy/sell an asset at a fixed price.
- **Forward Contract:** Agreement to buy/sell in the future at a pre-set price.

Why so many products? Each provides different ways to invest, manage risk, or speculate. Understanding these is essential for anyone working in finance, or just managing their own money.



7. Return and Risk: The Building Blocks

Return

$$\text{Return} = \frac{\text{New Price} - \text{Original Price}}{\text{Original Price}}$$

Example: Buy at \$100, sell at \$110:

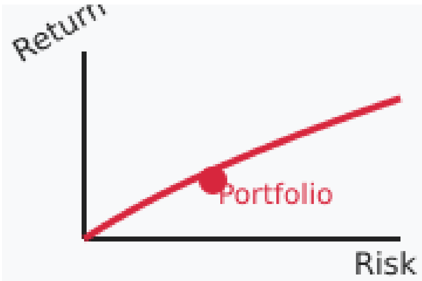
$$\text{Return} = \frac{110 - 100}{100} = 0.10 = 10\%$$

Interpretation: Return tells you how much you have gained (or lost) as a proportion of your original investment. It's the basic measure to compare different investments.

Risk and Volatility

Risk means the outcome is uncertain. **Volatility** (standard deviation of returns) is a common measure of risk.

Why is risk important? Most people prefer a steady, predictable return (like a savings account) to a risky one, unless the reward is bigger. Understanding and measuring risk helps you make better choices.



8. Portfolio, Risk, and Return: Understanding the Relationship

What is a Portfolio?

A **portfolio** is a collection of different financial assets (such as stocks, bonds, cash, or other investments) held by an individual or institution.

Why do we build portfolios?

- To spread investments across multiple assets (not "putting all eggs in one basket").
- To balance potential gains (returns) with the risks of losing money.
- To achieve specific financial goals, whether short-term or long-term.

What is Return?

- **Return** is the gain or loss on an investment over a specific period.
- Usually measured as a percentage of the initial investment.
- In a portfolio, the return is the weighted average of the returns of all assets in the portfolio.

What is Risk?

- **Risk** is the possibility that the actual returns will differ from the expected returns—often measured by the standard deviation or variance of returns.
- More uncertainty = higher risk.
- In portfolios, risk depends not only on the individual risks of the assets, but also on how the assets' returns move together (correlation).

Relationship between Portfolio, Risk, and Return

- A portfolio allows an investor to combine assets with different risks and returns.
- By diversifying (holding different assets), you can often **reduce overall risk** without proportionally reducing expected return.
- The key insight: the risk of a portfolio is **not just** the sum of the risks of its parts, but depends on how the returns of the assets are related.
- **Efficient portfolios** achieve the best possible return for a given level of risk.

Key Takeaways

- Portfolios are the building blocks of investment strategy.
- Diversification reduces risk—this is a central principle in finance.
- Portfolio risk and return depend on both asset selection and how those assets interact.
- Understanding this relationship is essential for managing investments and making rational financial decisions.

9. Random Variables in Finance: A Simple Example

Suppose a stock can go to:

$$S = \begin{cases} 120, & \text{with probability } 0.6 \\ 80, & \text{with probability } 0.4 \end{cases}$$

$$\mathbb{E}[S] = 0.6 \times 120 + 0.4 \times 80 = 104$$

$$\text{Expected Return} = \frac{104 - 100}{100} = 4\%$$

$$\text{Variance} = 0.6(120 - 104)^2 + 0.4(80 - 104)^2 = 384$$

$$\text{Standard Deviation} = \sqrt{384} \approx 19.6$$

What does this mean? The expected return is positive (good!), but the standard deviation (volatility) is nearly 20—so the outcome is quite uncertain. This is how math lets us "see" both the opportunity and the risk.

10. No-Arbitrage Principle

Arbitrage

Arbitrage: Risk-free profit.
Example: Buy gold in one market at \$1000, sell in another at \$1050. Profit = \$50, risk-free.

Why is this important? In real markets, opportunities for arbitrage (risk-free profit) are rare and disappear quickly—if they didn't, everyone would make easy money! The no-arbitrage principle is fundamental because it guides us to determine what a "fair price" for any asset or derivative should be.

Key Insight

If a financial product is priced too high or too low, traders will immediately capitalize on the discrepancy, buying from the cheaper market and selling to the more expensive market, until prices equalize. This process keeps the market efficient.

In summary: The no-arbitrage principle is a bedrock of financial mathematics—it tells us that prices in a fair and competitive market must not allow for free money. Most pricing models in finance, including those for options and other derivatives, are built to respect this principle.

Today's Goals

- Understand what an **option** is, in simple language and with technical definitions.
- Learn about **call** and **put** options, with clear examples.
- Grasp the meaning and importance of the **strike price**.
- Identify the main **market participants** in options markets.
- See analogies, formulas, and summary tables for clarity.

11. What is an Option?

Simple and Intuitive Definition

An **option** is a **financial contract** that gives the buyer the **right**, but **not the obligation**, to **buy or sell** an asset (like a stock) at a fixed **price** on or before a certain **date**.

Think of it like reserving something — you lock in a price now, but you only buy/sell if it benefits you later.

12. Types of Options: Call and Put

1. Call Option

Call Option

Gives the **right to BUY** an asset at a specified **strike price** before or at a certain date.

When would you use it? If you think the price will go **up**.

Example:

- You buy a call option to buy a stock at €100.
- If the market price rises to €120, you exercise your option: buy at €100, sell at €120.
- **Profit:** €20 (before considering the cost of the option).
- If the price stays below €100, you let the option expire. Your **maximum loss** is just the cost (premium) you paid.

2. Put Option

Put Option

Gives the **right to SELL** an asset at a specified **strike price** before or at a certain date.
When would you use it? If you think the price will go **down**.

Example:

- You buy a put option to sell a stock at €100.
- If the market price drops to €80, you exercise your option: sell at €100, buy in the market at €80.
- **Profit:** €20 (before considering the cost of the option).
- If the price stays above €100, you let the option expire. Your **maximum loss** is just the cost (premium) you paid.

13. The Strike Price (Exercise Price)

Definition: Strike Price (Exercise Price)

The **strike price** (also called the **exercise price**) is the **fixed price** at which the holder of an option can **buy** (for a call) or **sell** (for a put) the underlying asset.
It is **agreed in advance** as part of the option contract.

Simple Analogy:

Analogy: The Concert Ticket Coupon

Imagine you have a coupon that says:
*"You can buy a concert ticket for **€50** any time in the next month."*

- That **€50** is your **strike price**.
- Even if the ticket price becomes €100 later, your coupon lets you buy it at €50.
- The coupon has **value** — you can buy cheap and sell, or just enjoy the savings.

14. Financial Meaning of Strike Price

For Call Options

- The strike price is the price at which you can **buy** the asset.
- You profit when the **market price** is above the **strike price**:

$$\text{Market Price} > \text{Strike Price}$$

For Put Options

- The strike price is the price at which you can **sell** the asset.
- You profit when the **market price** is below the **strike price**:

$$\text{Market Price} < \text{Strike Price}$$

15. Formulas and Numerical Example

Call Option Payoff:

$$\text{Payoff}_{\text{call}} = \max(\text{Market Price} - \text{Strike Price},\ 0)$$

Put Option Payoff:

$$\text{Payoff}_{\text{put}} = \max(\text{Strike Price} - \text{Market Price},\ 0)$$

Worked Example: Call Option

Suppose you buy a **call option**:

- Strike Price = €100
- Market Price later = €120

Your payoff is:

$$\text{Payoff} = \max(120 - 100, 0) = 20$$

16. Summary Table: Call vs. Put

Feature	Call Option	Put Option
Right to...	Buy an asset	Sell an asset
Use it when...	Price goes **up**	Price goes **down**
Profit when...	Market > Strike	Market < Strike
Loss Limited?	Yes, to premium paid	Yes, to premium paid

17. Who are the Market Participants in Options?

Key Market Participants

- **Buyers (Holders):** Purchase options for speculation, hedging, or investment.
- **Sellers (Writers):** Write (sell) options, earn the premium, and take on the obligation.
- **Speculators:** Try to profit from price movements in the underlying asset by buying or selling options.
- **Hedgers:** Use options to reduce risk (for example, a farmer using options to lock in a selling price for crops).
- **Arbitrageurs:** Seek to exploit pricing inefficiencies between markets.
- **Market makers:** Provide liquidity by quoting both buy and sell prices for options.

Each participant has a different goal: *risk management, profit, providing liquidity, or balancing portfolios.*

18. Key Takeaways

- An **option** gives you flexibility: you decide later whether to use it!
- **Call option**: Right to buy; profit if market price rises above strike.
- **Put option**: Right to sell; profit if market price falls below strike.
- The **strike price** is agreed in advance and is central to every option contract.
- Different **market participants** interact in the options market for different reasons.
- **Maximum loss** is always the amount paid for the option (the premium).
- The value of an option depends on the relationship between the market price and the strike price.

19. European vs. American Options

What are European and American Options?

There are two main styles of financial options: **European** and **American**. Both give the holder special rights, but differ in **when** those rights can be used.

European Option

Definition: European Option

A **European option** can be **exercised only at maturity**, i.e., only on the expiration date.

Example:

- You buy a **European call option** on a stock with
  - Strike price = €100
  - Expiration = 3 months
- You can decide whether to exercise the option only **on the final day** (in 3 months), not before.

Mathematical Implication

Easier to price using models like the **Black-Scholes** or **Binomial Tree**, since the exercise decision is only at a single point in time.

American Option

Definition: American Option

An **American option** can be **exercised at any time up to and including expiration**.

Example:

- You buy an **American put option** with
  - Strike price = €80
  - Expiration = 1 month
- You can choose to exercise it **today**, tomorrow, or any day before it expires.

Mathematical Implication

More complex to model, because we need to check at **each time step** whether early exercise is better than holding the option.

Summary Comparison

Feature	European Option	American Option
Exercise Time	Only at maturity	Any time up to maturity
Flexibility	Less flexible	More flexible
Pricing Complexity	Easier	More complex
Typically More Expensive	No	Yes (due to flexibility)

Did You Know?

Most traded stock options in the USA are **American**, while many index options and most options traded in Europe are **European**.

Want to see a visual example? Using a binomial tree, you can see exactly when it might make sense to exercise an American option early, while European options can only be exercised at the end.

Practice Questions

1. If you buy a call option on a stock with a strike price of €50 and the market price rises to €60, what is your profit (ignoring the premium)?
2. If you buy a put option on a stock with a strike price of €70 and the market price falls to €60, what is your profit (ignoring the premium)?
3. If the market price stays below the strike price for a call option, what happens?
4. For a put option with a strike price of €80 and the market price falls to €70, what is your payoff?
5. If you have a call option with a strike price of €40 and the market price rises to €38, what is your payoff?
6. Why is it important to agree on the strike price **before** the market moves?
7. Name two types of market participants in options markets and explain their roles.

20. Real-World Applications of Mathematical Finance

Let's see how the concepts from today's lecture apply to real-life financial decisions.

1. Portfolio Allocation Example

Portfolio Allocation

Problem: You have €100,000 to invest in:

- Risky stock (expected return: 10%, std: 20%)
- Risk-free bond (return: 4%)

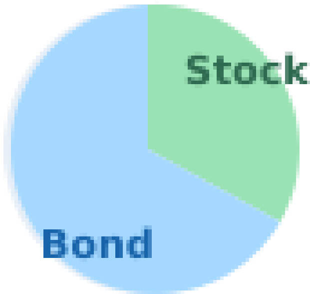
If portfolio risk limit is 10%:

$$x \cdot 20\% \leq 10\% \implies x \leq 0.5$$

So, invest up to 50% in stock.
Expected portfolio return:

$$R_p = 0.5 \times 10\% + 0.5 \times 4\% = 7\%$$

Mathematics helps us optimize investment mix.



Explanation: You can use mathematics to balance risk and return, and to decide the best mix of investments for your personal goals and risk tolerance. This is exactly how investment managers build portfolios for their clients!

2. Insurance Pricing Example

Insurance Pricing

Problem: Insuring a house worth €500,000. Fire probability: 0.1%.

$$\mathbb{E}[\text{Loss}] = 0.001 \times 500,000 = £500$$

With a 20% markup: charge €600 for the policy.
Conclusion: Probability ensures fair and profitable pricing.

Further explanation: This is how insurance companies set their prices: by calculating the expected loss using probability, then adding a margin for profit and expenses.

3. Risk-Neutral Pricing of a Game

Game:

- Win €100 (probability 0.5)
- Lose €50 (probability 0.5)

$$\mathbb{E}[X] = 0.5 \times 100 + 0.5 \times (-50) = 25$$

Fair price: €25. In finance, we often use *risk-neutral valuation* (details to come).
What does this mean? If you can play this game for less than €25, it's a good deal; above €25, it's not. This is the same principle used for buying and selling financial products.

4. Loan Default Risk Modeling

- Loan: €100,000

- Default probability: 2%
- Recovery: €20,000

$$\text{Expected Loss} = 0.02 \times (100,000 - 20,000) = £1,600$$

Banks use this to set interest rates.
Explanation: Banks assess the risk of lending by estimating expected losses due to defaults, and then set interest rates high enough to cover these losses and make a profit.

5. Derivatives Hedging Example

A company worried about raw material price drops buys a *put option* to protect itself. Mathematical finance helps decide:

- How much protection to buy
- What is a fair price
- What's the remaining risk

Why is this important? Derivatives allow companies and investors to manage risks they can't control—like commodity prices, exchange rates, or interest rates. Knowing how to price and use these instruments is a key skill in today's world.

21. Summary of Today's Key Points

Summary

- Finance is about making decisions under uncertainty.
- Probability is essential for modelling financial outcomes.
- Concepts like expectation, variance, and arbitrage are foundational.
- Simple models (like binomial) are powerful tools for pricing and risk management.

In essence: You have now seen that mathematics allows us to quantify uncertainty, estimate risks, and avoid costly mistakes. Even basic concepts like expected value and arbitrage are powerful in understanding how financial markets work.
Take-home message: With mathematical tools, you are better equipped to make rational, profitable decisions—not just in finance, but in any uncertain environment.

Takeaways and Next Steps

Key Takeaways

- Mathematical finance is everywhere in the real world.
- Probability and expectation are at the heart of pricing, investment, and risk.
- This course will teach you to use these tools to make smarter decisions.

Next Lectures:

- Building formal models like the binomial tree
- Option pricing formulas
- Simulations using programming

No matter your background, we will work through all ideas step by step. Your curiosity and questions are welcome!

Homework / Practice Problems

Why this matters: Every concept in this assignment is a building block for more advanced topics like option pricing, hedging, and risk management. If you get comfortable with these basics, the advanced ideas will become much clearer. *Try solving by hand first—understanding the logic is more important than memorizing formulas. Bring your questions to class!*

1. Calculate the expected return for a portfolio with 30% in stock (10% return, 20% std) and 70% in bond (4% return).
2. Create your own game with uncertain rewards, and compute its fair price.
3. If the loan default rate rises to 5%, what is the new expected loss (with same loan and recovery terms)?

Bring your questions and solutions to the next class!