Exercise Sheet: Lecture 3 – Brownian Motion and Itô Calculus

Dr. Abhishek Chaudhary

Numerical Analysis Group, Department of Mathematics,

University of Tübingen

Lecture 3: Advanced Exercises

This exercise sheet covers key concepts and deeper understanding for Brownian motion, stochastic calculus, the Itô integral, and stochastic differential equations. Attempt all problems for mastery.

Brownian Motion: Theory and Computation

1. **[Computation]** (*Simulation*)
   Describe how to simulate a path of a one-dimensional Wiener process numerically on $[0, T]$ with step size h. Write pseudocode or a short routine (in Python/Matlab/R) for one simulated path.

2. **[Theory]** (*Total Variation*)
   Let W_t be a standard Brownian motion. Show that with probability one, the sample path $t \mapsto W_t$ has unbounded total variation on any interval $[a, b]$.

3. **[Theory]** (*Quadratic Variation*)
   Show that for any partition P_N of $[0, T]$ with mesh tending to zero,

   $$\sum_{n=1}^{N} (W_{t_n} - W_{t_{n-1}})^2 \to T$$

   in $L^2(\mathbb{P})$. What does this imply about the "roughness" of Brownian motion compared to smooth functions?

4. **[Computation]** (*Covariance Function*)
   Show that for standard Brownian motion, $\mathbb{E}[W_s W_t] = \min(s, t)$ for $s, t \geq 0$.

5. **[Theory]** (*Markov Property*)
   Prove that Brownian motion is a Markov process, i.e., for $0 \leq s < t$, the future increment $W_t - W_s$ is independent of the past σ-algebra $\mathcal{F}_s$.

6. **[Computation]** (*Distribution of Increments*)
   Let $0 \leq s < t$. Show that $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$, and that increments over non-overlapping intervals are independent.

Itô Integral and Stochastic Calculus

7. **[Theory]** (*Itô Isometry for Elementary Functions*)
   Let ϕ be an elementary process. Prove that

$$\mathbb{E}\left(\left(\int_0^T \phi(t)\, dW_t\right)^2\right) = \mathbb{E}\left(\int_0^T \phi^2(t)\, dt\right).$$

8. **[Computation]** (*Itô Integral of W with respect to W*)
   Compute $\int_0^t W(s)\, dW(s)$ and show that

$$\int_0^t W(s)\, dW(s) = \frac{1}{2}W^2(t) - \frac{1}{2}t.$$

9. **[Theory]** (*Itô vs Riemann–Stieltjes*)
   For a deterministic, continuously differentiable function $v : [0, t] \to \mathbb{R}$ with $v(0) = 0$, show that

$$\int_0^t v(s)\, dv(s) = \frac{1}{2}v^2(t)$$

   and explain the difference with the Itô formula for W.

10. **[Computation]** (*Martingale Property*)
    Show that for any $t \geq 0$, $\mathbb{E}[W_t | \mathcal{F}_s] = W_s$ for $0 \leq s \leq t$. Conclude that Brownian motion is a martingale.

11. **[Theory]** (*Adapted Processes*)
    Define what it means for a stochastic process to be adapted to a filtration. Give an example and a non-example.

12. **[Theory]** (*Linearity and Zero Mean of Itô Integral*)
    Let u, v be adapted processes, $c \in \mathbb{R}$. Show:

$$\int_0^T (cu(t) + v(t))\, dW_t = c \int_0^T u(t)\, dW_t + \int_0^T v(t)\, dW_t$$

and $\mathbb{E}\left(\int_0^T u(t)\,dW_t\right) = 0$.

13. **[Computation]** (*SDE for Geometric Brownian Motion*)
Write down the SDE for geometric Brownian motion $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ and find its explicit solution.

14. **[Theory]** (*Itô Formula*)
Let X_t solve $dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t$ and $F \in C^{1,2}$. Write the Itô formula for $F(t, X_t)$ and explain the meaning of the second-order term.

15. **[Computation]** (*Itô Formula Example*)
Let $X_t = W_t$, $F(x) = x^4$. Use Itô's formula to compute $dF(X_t)$ and, by taking expectations, compute $\mathbb{E}[W_t^4]$.

Submission Instructions

Write clear and complete solutions. You may discuss with classmates, but the work you submit must be written by you. Please submit before 17th June 2025.