Exercise Sheet: Proof of Itô's Formula

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Advanced Itô Calculus – Proof Exercises

This exercise sheet focuses on filling in the rigorous details of the proof of Itô's formula for general Itô processes. Provide complete arguments and justifications for all steps.

Itô's Formula: Rigorous Steps

Consider an Itô process

$$X_t = X_0 + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dW_s.$$

Below are five exercises designed to fill in the omitted steps in the proof of Itô's formula for $F(t, X_t)$.

1. [**Analysis**] (*Approximation of the coefficient function F*)
   Let $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable (possibly with unbounded derivatives). Show that there exists a sequence $(F_m)_{m \geq 1} \subset C^2([0, \infty) \times \mathbb{R})$ with each F_m having bounded partial derivatives and, for every compact set $K \subset [0, \infty) \times \mathbb{R}$,

   $$\sup_{(t,x) \in K} |F_m(t, x) - F(t, x)| + \sup_{(t,x) \in K} \sum_{|\alpha| \leq 2} |\partial^\alpha F_m(t, x) - \partial^\alpha F(t, x)| \xrightarrow{m \to \infty} 0.$$

   (Here α is a multi–index, ∂^α denotes any partial derivative up to order 2.) Outline a construction using mollifiers and justify uniform convergence on compacts.

2. [**Probability**] (*Elementary approximations of the processes f and g*)

Recall that an *elementary adapted process* has the form

$$H(t,\omega) = \sum_{n=0}^{N-1} A_n(\omega)\, 1_{[t_n,t_{n+1})}(t), \quad 0 = t_0 < t_1 < \cdots < t_N = T,$$

where each A_n is $\mathcal{F}_{t_n}$–measurable.

(a) Show how to approximate the adapted processes $t \mapsto f(t, X_t)$ and $t \mapsto g(t, X_t)$ by elementary processes in $L^2([0,T] \times \Omega)$.

(b) Deduce that

$$\int_0^T f(s, X_s)\, ds - \sum_{n=0}^{N-1} f(t_n, X_{t_n})(t_{n+1} - t_n) \xrightarrow{L^2} 0$$

and similarly for the stochastic integral with respect to W_s, as the mesh $\max_n(t_{n+1} - t_n) \to 0$.

3. [**Analysis/Probability**] (*Convergence of the main Riemann sums*)
Define the partial sums

$$S_N^{(1)} = \sum_{n=0}^{N-1} \partial_t F(t_n, X_{t_n})(t_{n+1} - t_n),$$
$$S_N^{(2)} = \sum_{n=0}^{N-1} \partial_x F(t_n, X_{t_n}) f(t_n, X_{t_n})(t_{n+1} - t_n),$$
$$S_N^{(3)} = \sum_{n=0}^{N-1} \partial_x F(t_n, X_{t_n}) g(t_n, X_{t_n})(W_{t_{n+1}} - W_{t_n}).$$

Prove in $L^2(\Omega)$ that as $N \to \infty$,

$$S_N^{(1)} \to \int_0^T \partial_t F(s, X_s)\, ds, \quad S_N^{(2)} \to \int_0^T \partial_x F(s, X_s) f(s, X_s)\, ds, \quad S_N^{(3)} \to \int_0^T \partial_x F(s, X_s) g(s, X_s)\, dW$$

Use the L^2-isometry for deterministic Riemann sums and the Itô isometry.

4. [**Probability**] (*Negligibility of mixed–order and remainder terms*)
In the Taylor expansion of $F(t_{n+1}, X_{t_{n+1}})$ around (t_n, X_{t_n}), one encounters terms

$$T_n^1 = \partial_t \partial_x F(t_n, X_{t_n})(t_{n+1} - t_n)(X_{t_{n+1}} - X_{t_n}), \quad T_n^2 = \tfrac{1}{2} \partial_t^2 F(t_n, X_{t_n})(t_{n+1} - t_n)^2, \quad r_n = \text{higher-order}$$

Show that each of

$$\sum_{n=0}^{N-1} T_n^1, \quad \sum_{n=0}^{N-1} T_n^2, \quad \sum_{n=0}^{N-1} r_n$$

converges to zero in $L^2(\Omega)$ as $N \to \infty$, using boundedness of derivatives and moment estimates for $X_{t_{n+1}} - X_{t_n} = O_p(\sqrt{t_{n+1} - t_n})$.

5. **[Probability]** (*Emergence of the quadratic variation term*)
   Let

$$Q_N = \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F(t_n, X_{t_n}) g(t_n, X_{t_n})^2 \left[(W_{t_{n+1}} - W_{t_n})^2 - (t_{n+1} - t_n) \right].$$

   (a) Show that $\mathbb{E}[Q_N] = 0$ for every N.

   (b) Using independence and moments of the Gaussian increments, prove

$$\mathbb{E}[Q_N^2] \longrightarrow 0 \qquad \text{as } N \to \infty.$$

   (c) Conclude that

$$\frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F(t_n, X_{t_n}) g(t_n, X_{t_n})^2 (W_{t_{n+1}} - W_{t_n})^2 \xrightarrow{L^2} \frac{1}{2} \int_0^T \partial_x^2 F(s, X_s) g(s, X_s)^2 \, ds.$$

Submission Instructions

Write clear and complete solutions. You may discuss with classmates, but the work you submit must be written by you. Please submit before 10th July 2025.