Exercise Sheet: Solving Black-Scholes Equations by Monte Carlo Methods

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Assignment Overview

This exercise sheet explores the use of Monte Carlo simulation to solve the Black-Scholes equation for pricing European and path-dependent options. Through these exercises, you will simulate asset price paths, compute option payoffs, and analyze the accuracy of your estimates. For each exercise:

- Use the specified parameters unless otherwise noted.
- Implement your simulations in a programming language of your choice (e.g., Python, MATLAB, R).
- Submit your code, results, and plots with your solutions.

Note: For reproducibility, set a random seed when generating random numbers, especially when comparing results across different sample sizes.

Exercises

Exercise 1. Simulate Geometric Brownian Motion

Simulate $M = 10,000$ paths of the asset price $S(T)$ at time $T = 1$ year, given the Black-Scholes SDE under the risk-neutral measure:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0,$$

with parameters $r = 0.05$, $\sigma = 0.2$, $S_0 = 100$.

(a) Use the analytical solution:

$$S(T) = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma W(T)\right],$$

where $W(T) \sim \mathcal{N}(0, T)$. Generate $W(T) = \sqrt{T} Z$, with $Z \sim \mathcal{N}(0, 1)$.

(b) Plot a histogram of the simulated $S(T)$ values to visualize the distribution.

(c) Compute the sample mean and standard deviation of $S(T)$. Compare the sample mean to the theoretical expectation $S_0 e^{rT}$. **Hint**: $S(T)$ follows a log-normal distribution, and its theoretical mean is $S_0 e^{rT}$. A close match indicates a correct simulation.

Exercise 2. Monte Carlo Pricing of a European Call Option

Price a European call option with strike $K = 110$ and maturity $T = 1$ year using the simulated paths from Exercise 1.

(a) For each simulated $S(T)$, compute the payoff:

$$\psi(S(T)) = \max(S(T) - K, 0).$$

(b) Compute the Monte Carlo estimate of the option price:

$$C_{\mathrm{MC}} = e^{-rT} \cdot \frac{1}{M} \sum_{i=1}^{M} \psi(S(T, \omega_i)).$$

(c) Report your estimated price.

(d) (Optional) Compare your result to the Black-Scholes formula for a European call option. **Hint**: Use the same $S(T)$ paths from Exercise 1 to maintain consistency.

Exercise 3. Effect of Sample Size on Accuracy

Investigate how the number of Monte Carlo samples M affects the accuracy of the option price estimate from Exercise 2.

(a) Repeat Exercise 2 for $M = 100$, $1,000$, $10,000$, and $100,000$.

(b) For each M:

- Report the price estimate C_{MC}.
- Compute the standard error: $\mathrm{SE} = \frac{\text{sample standard deviation of } \psi(S(T))}{\sqrt{M}}$.

(c) Plot the estimated price with error bars (using SE) versus M on a log scale for M.

(d) Briefly explain the observed convergence and error trends. **Hint**: The standard error should decrease as $1/\sqrt{M}$, and the estimate should stabilize as M increases.

Exercise 4. Pricing a European Put Option

Price a European put option with strike $K = 90$, using the same parameters as in Exercise 1.

(a) For each simulated $S(T)$, compute the put payoff:

$$\psi_{\text{put}}(S(T)) = \max(K - S(T), 0).$$

(b) Compute the Monte Carlo estimate:

$$P_{\text{MC}} = e^{-rT} \cdot \frac{1}{M} \sum_{i=1}^{M} \psi_{\text{put}}(S(T, \omega_i)).$$

(c) Report your result.

(d) (Optional) Compare your result to the Black-Scholes formula for a European put option.

Exercise 5. Path-dependent Option – Asian Call (Arithmetic Average)

Simulate and price an Asian call option with payoff based on the arithmetic average of the asset price:

$$\psi_{\text{Asian}} = \max\left(\frac{1}{N} \sum_{n=1}^{N} S(t_n) - K,\, 0\right),$$

where $K = 100$, $T = 1$, and $N = 50$ time steps.

(a) Discretize $[0, T]$ into $N = 50$ equal steps, with $\Delta t = T/N$.

(b) For each path, simulate $S(t_n)$ at each time step using:

$$S_{n+1} = S_n \exp\left[\left(r - \frac{1}{2}\sigma^2\right) \Delta t + \sigma \sqrt{\Delta t} Z_n\right],$$

where $Z_n \sim \mathcal{N}(0, 1)$.

(c) For each path, compute the arithmetic average $\frac{1}{N} \sum_{n=1}^{N} S(t_n)$ and the payoff ψ_{Asian}.

(d) Estimate the option price:

$$C_{\text{Asian, MC}} = e^{-rT} \cdot \frac{1}{M} \sum_{i=1}^{M} \psi_{\text{Asian}}(\omega_i).$$

(e) Qualitatively compare the Asian option price to the European call price from Exercise 2. Which is typically higher? Why? **Hint**: Asian options depend on the path average, which is less volatile than the final price, often leading to a lower price than a European call.

Submission Instructions

Submit a report containing:

- Your code for each exercise.
- Results, plots, and comparisons as requested.
- Brief explanations of your observations, particularly for Exercises 3 and 5.

Collaboration is allowed, but your submission must be your own work. This is optional.