Der Sinus an der Wiege der Naturwissenschaften

G. Wanner (“Webinar” Genf ⇒ Tübingen, Feb. 2021)

“Wiege ??” ... das beginnt mit etwas ganz Altem ...

A.H.Rhind
(1833–1863)

A.Eisenlohr

Rhind Papyrus, 1950 v.Chr. ⇒ 1650 v.Chr..

A.H.Rhind ⇒ August Eisenlohr (1877):

Eisenlohr nr.

Fl. Rechteck 49

Fl. Kreis 50

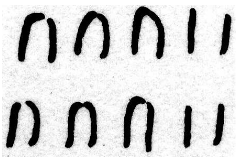
Fl. Dreieck 51

Fl. Trapez 52

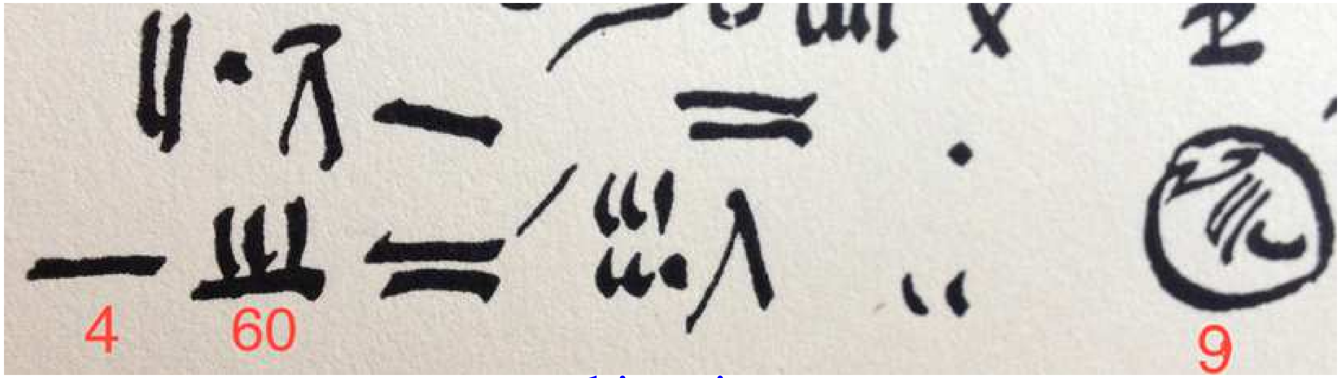
Feldteilung 53

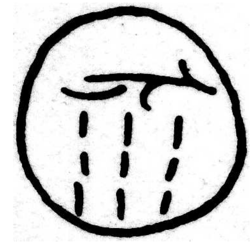
Rhind Nr. 50: **Kreisfl.** $= \left(\text{diam} - \frac{\text{diam}}{9}\right)$ **zum Quadrat**

T.E.Peet (1923)



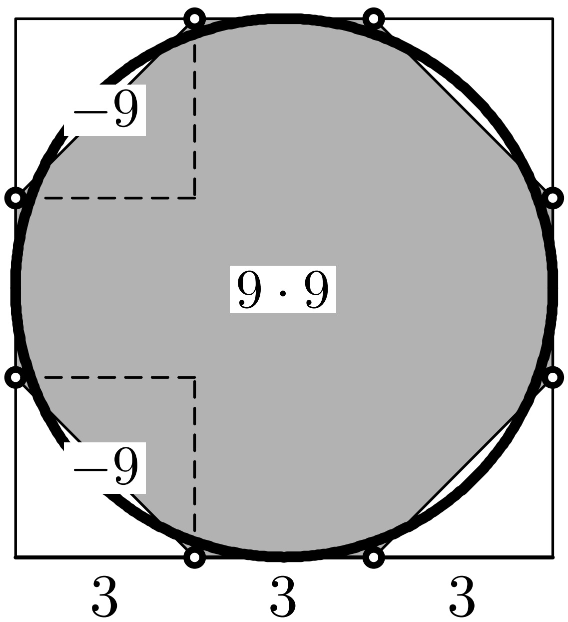
hieroglyphic

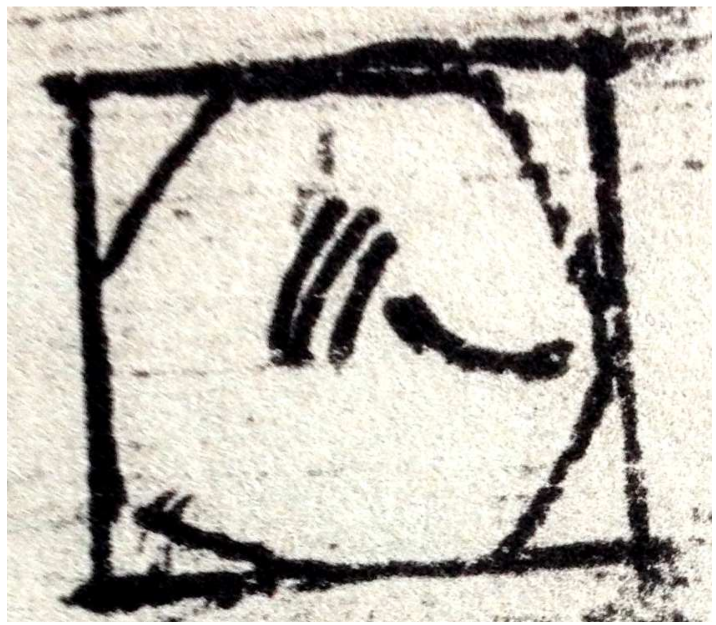




hieratic

Warum? Rhind Nr. 48:

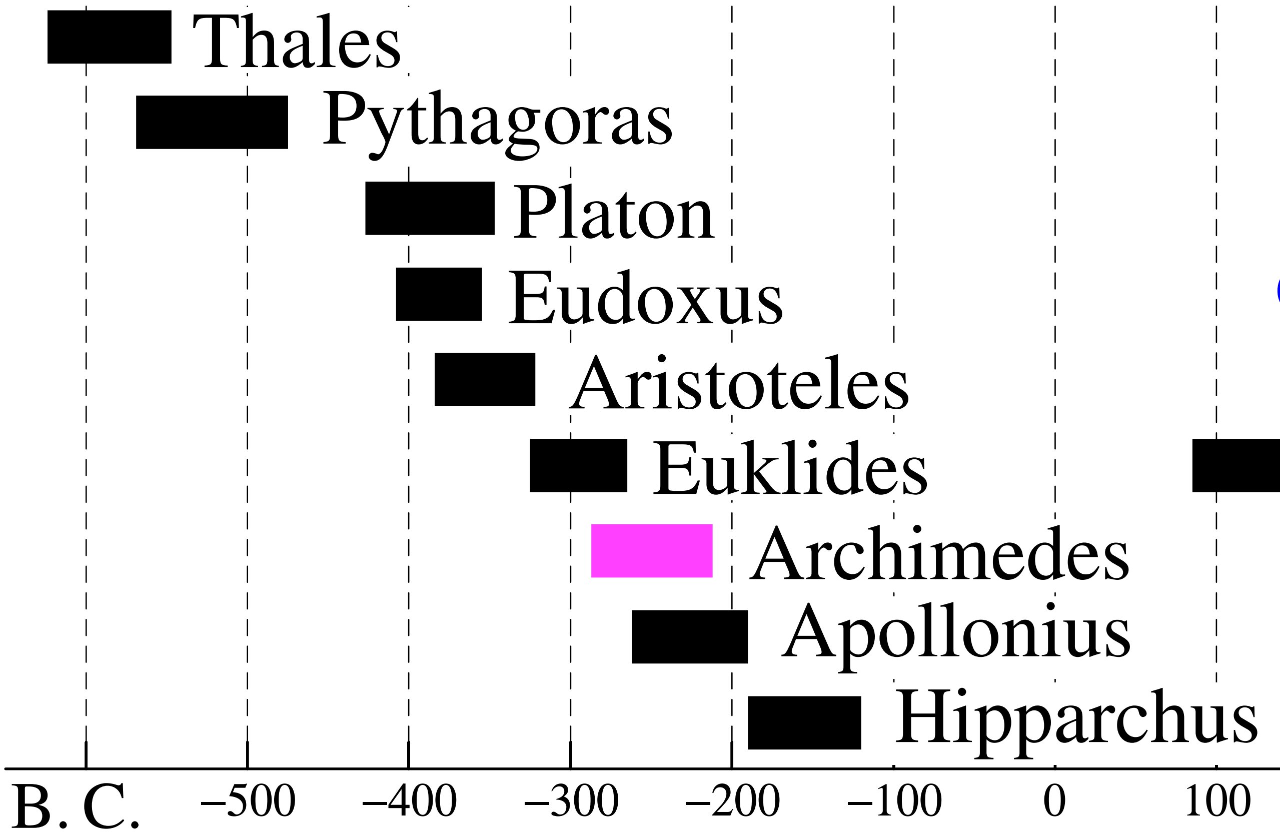
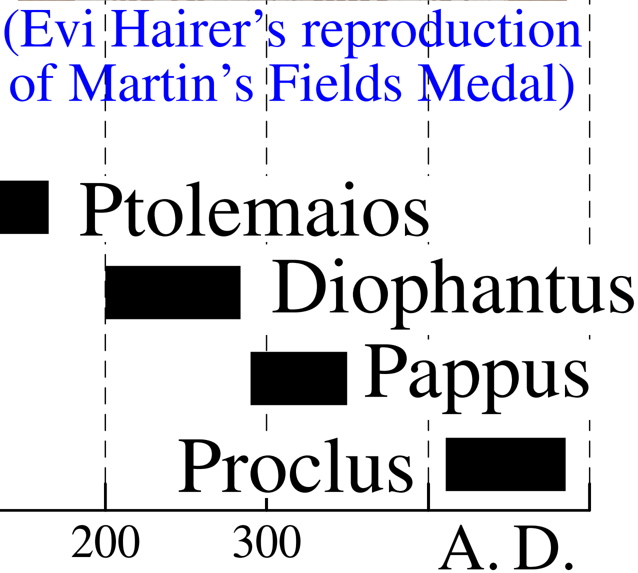




$$\text{Fläche } = 9 \cdot 9 - 2 \cdot 9 = 63 \quad \text{(ist kein Quadrat)} \quad \approx 64 = 8^2$$

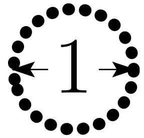
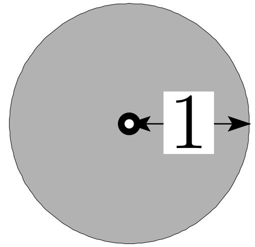
(exakt ????? NEIN!)

Die griechische Strenge: Archimedes

(Evi Hairer’s reproduction of Martin’s Fields Medal)

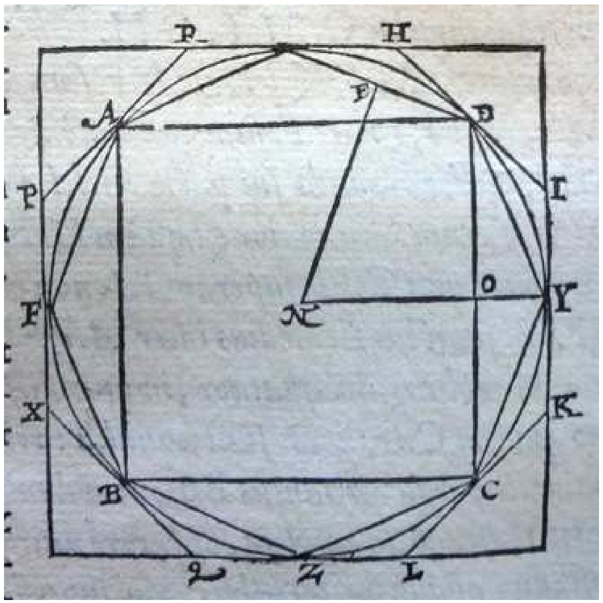
(Archimedes, “Kreismessung”, impr. 1665)

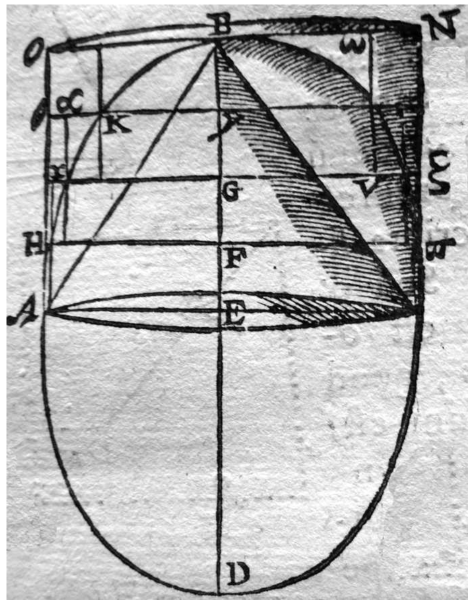
Suchen π $= \pi$erimeter  $=$ Fl. 

W. Jones 1706

Euklid (Buch V): Einzig **streng** ist: $\boxed{r < \pi < R}$ r, R **rat.**

Archimedes “Riemannsche Unter- und Obersummen”:

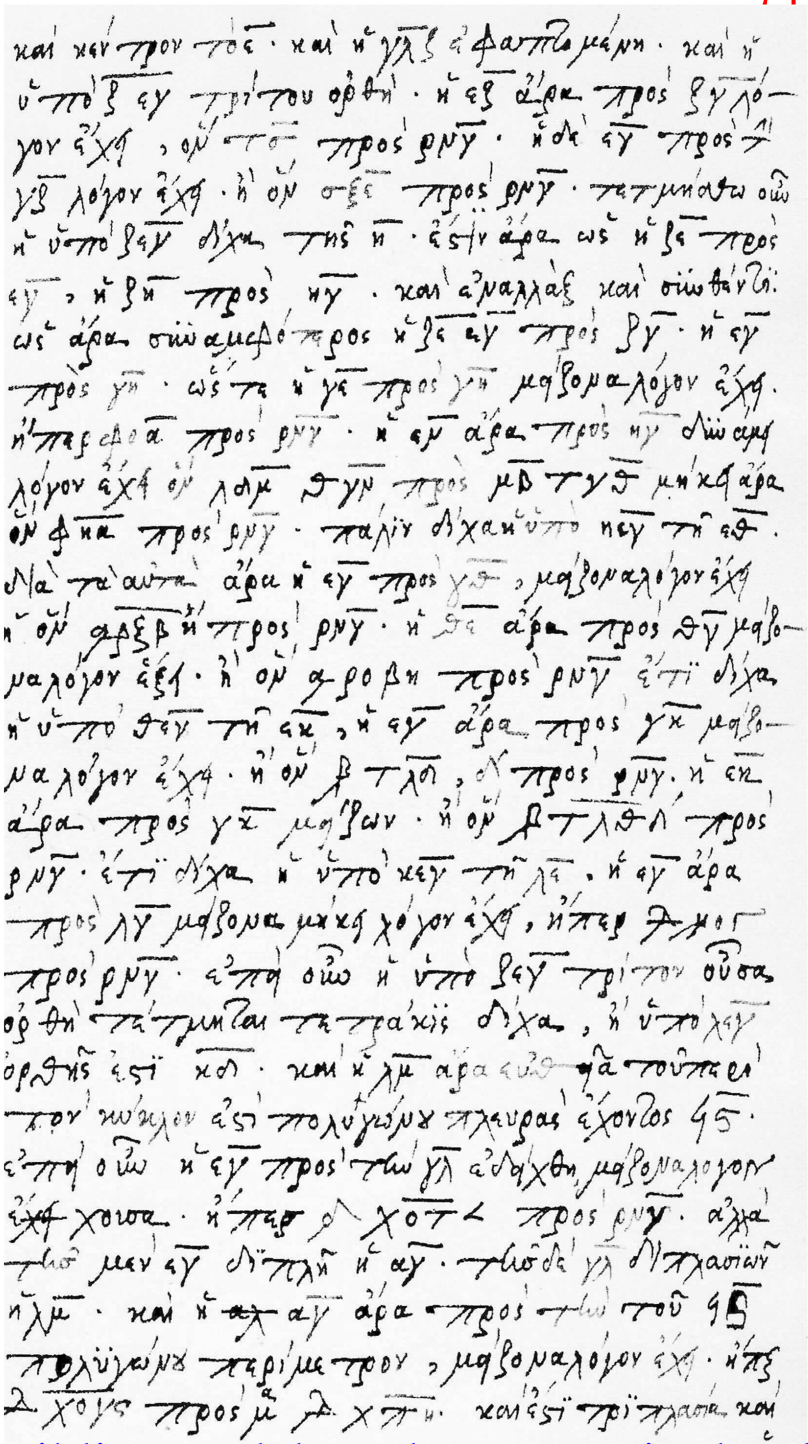


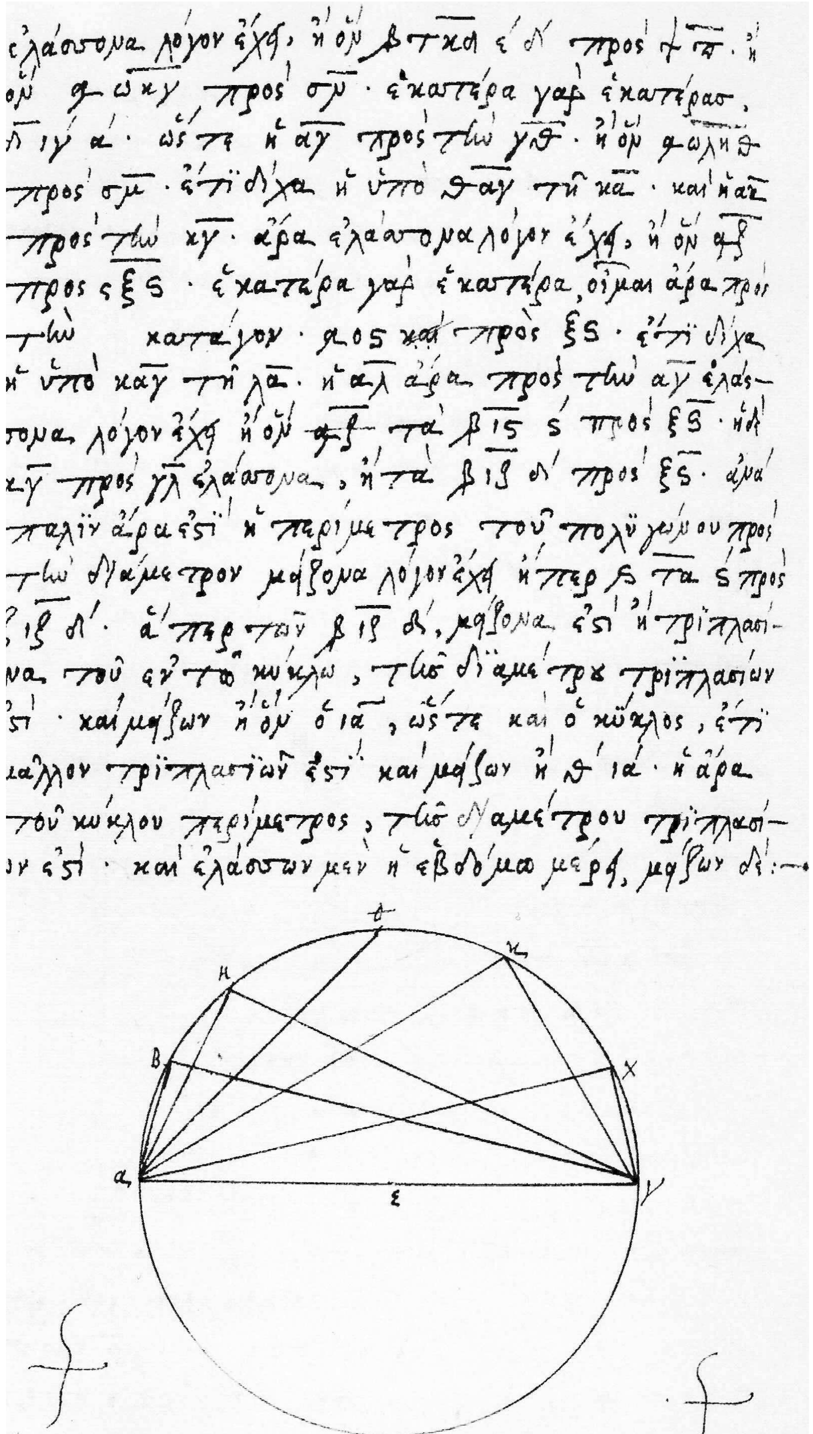


Proposition 3: $(3.1408 <)$ $\boxed{3\frac{10}{71} < \pi < 3\frac{1}{7}}$ (< 3.1428572)

Rhind: $\frac{4 \cdot 64}{81} = 3.1605$

Archimedes' Beweis für $3\frac{10}{71} < \pi$:





Biblioteca del Real Monasterio de El Escorial, manuscrito X-I-14 (©Ph. Henry) d'où : $\frac{A\Gamma}{A\Gamma + AB} = \frac{\Gamma Z}{\Gamma Z + ZB}$, ou $\frac{A\Gamma}{\Gamma Z} = \frac{A\Gamma + AB}{B\Gamma}$, d'où : $\frac{AH}{H\Gamma} = \frac{A\Gamma + AB}{B\Gamma} = \frac{A\Gamma}{B\Gamma} + \frac{AB}{B\Gamma}$. Or, $\frac{A\Gamma}{B\Gamma} = \frac{1560}{780}$, et $\frac{AB}{B\Gamma} < \frac{1351}{780}$; donc $\frac{AH}{H\Gamma} < \frac{1560 + 1351}{780}$, ou $\frac{AH}{H\Gamma} < \frac{2911}{780}$. D'autre part, $\frac{\overline{AH}^2}{\overline{H\Gamma}^2} < \frac{(2911)^2}{(780)^2}$, d'où : $\frac{\overline{AH}^2 + \overline{H\Gamma}^2}{\overline{H\Gamma}^2} < \frac{2911^2 + 780^2}{780^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{H\Gamma}^2} < \frac{9082321}{608400}$, d'où, comme le texte : $\frac{A\Gamma}{H\Gamma} < \frac{3013\frac{3}{4}}{780}$.

1. On aura, comme dans le cas précédent : $\frac{A\Theta}{\Theta\Gamma} = \frac{A\Gamma + AH}{H\Gamma} = \frac{A\Gamma}{HA} + \frac{AH}{H\Gamma}$, d'où, substituant les valeurs de ces deux derniers termes : $\frac{A\Theta}{\Theta\Gamma} < \frac{3013\frac{3}{4} + 2911}{780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{5924\frac{3}{4}}{780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{\frac{4}{13} \times 5924\frac{3}{4}}{\frac{4}{13} \times 780}$, ou $\frac{A\Theta}{\Theta\Gamma} < \frac{1823}{240}$. D'autre part, $\frac{\overline{A\Theta}^2}{\overline{\Theta\Gamma}^2} < \frac{1823^2}{240^2}$, d'où $\frac{\overline{A\Theta}^2 + \overline{\Theta\Gamma}^2}{\overline{\Theta\Gamma}^2} < \frac{1823^2 + 240^2}{240^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{\Theta\Gamma}^2} < \frac{3380929}{57600}$, d'où, comme le texte : $\frac{A\Gamma}{\Theta\Gamma} < \frac{1838\frac{9}{11}}{240}$.

2. On aura de même : $\frac{AK}{K\Gamma} = \frac{A\Gamma + A\Theta}{\Theta\Gamma} = \frac{A\Gamma}{\Theta\Gamma} + \frac{A\Theta}{\Theta\Gamma}$. et, par substitution des valeurs trouvées pour ces deux derniers termes, il vient : $\frac{AK}{K\Gamma} < \frac{1838\frac{9}{11} + 1823}{240}$, ou $\frac{AK}{K\Gamma} < \frac{3661\frac{9}{11}}{240}$, ou $\frac{AK}{K\Gamma} < \frac{\frac{11}{40} 3661\frac{9}{11}}{\frac{11}{40} 240}$, ou $\frac{AK}{K\Gamma} < \frac{1007}{66}$. D'autre part, $\frac{\overline{AK}^2}{\overline{K\Gamma}^2} < \frac{1007^2}{66^2}$, d'où : $\frac{\overline{AK}^2 + \overline{K\Gamma}^2}{\overline{K\Gamma}^2} < \frac{1007^2 + 66^2}{66^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{K\Gamma}^2} < \frac{1018405}{14356}$, d'où, comme le texte : $\frac{A\Gamma}{K\Gamma} < \frac{1009\frac{1}{6}}{66}$.

3. On aura de même: $\frac{A\Lambda}{\Lambda\Gamma} = \frac{A\Gamma + AK}{K\Gamma} = \frac{A\Gamma}{K\Gamma} + \frac{AK}{K\Gamma}$, et, par substitution des valeurs précédentes : $\frac{A\Lambda}{\Lambda\Gamma} < \frac{1009\frac{1}{6}}{66} + \frac{1007}{66}$, ou $\frac{A\Lambda}{\Lambda\Gamma} < \frac{2016\frac{1}{6}}{66}$. D'autre part, $\frac{\overline{A\Lambda}^2 + \overline{\Lambda\Gamma}^2}{\overline{\Lambda\Gamma}^2} < \frac{(2016\frac{1}{6})^2 + 66^2}{66^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{\Lambda\Gamma}^2} < \frac{4069284\frac{1}{36}}{4356}$, d'où, comme le texte: $\frac{A\Gamma}{\Lambda\Gamma} < \frac{2017\frac{1}{4}}{66}$.

1. Sous-entendu : περίμετρος, le périmètre (du cercle).

2. La relation de la note avant-précédente donne, par inversion : $\frac{\Lambda\Gamma}{A\Gamma} > \frac{66}{2017\frac{1}{4}}$, d'où, observant que $96 \times \Lambda\Gamma$ = périmètre polygone inscrit de 96 côtés :

$$\frac{\text{périmètre polygone de 96 côtés}}{\text{diamètre cercle}} > \frac{96 \times 66}{2017\frac{1}{4}}, \text{ ou } > \frac{6336}{2017\frac{1}{4}}. \text{ Or, } \frac{6336}{2017\frac{1}{4}} > 3\tfrac{10}{71},$$

d'où périmètre polygone de 96 côtés $> 3\frac{10}{71}$ diamètre cercle, d'où, à fortiori, suivant le texte : Circonférence cercle $> 3\frac{10}{71}$ diamètre.

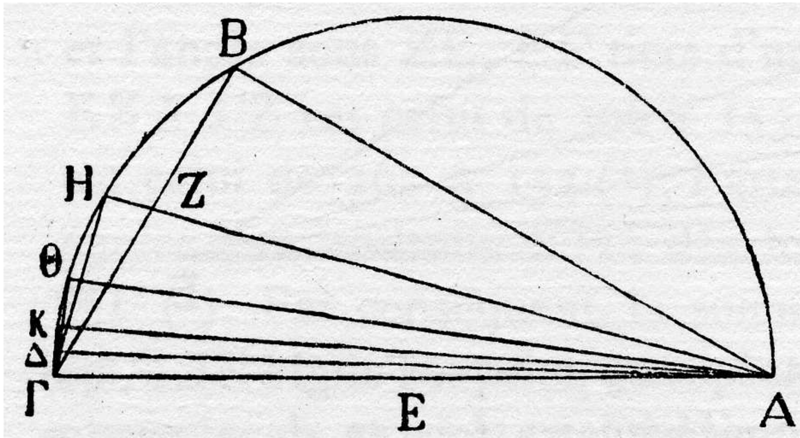
Entziffern ...

F. Commandino (grec-latin) 1558

Heiberg (grec) 1888

Th. Heath (english) 1920

⇐ Ver Eecke (français) 1921



$B\Gamma = \frac{1}{2}$ (6-gon)

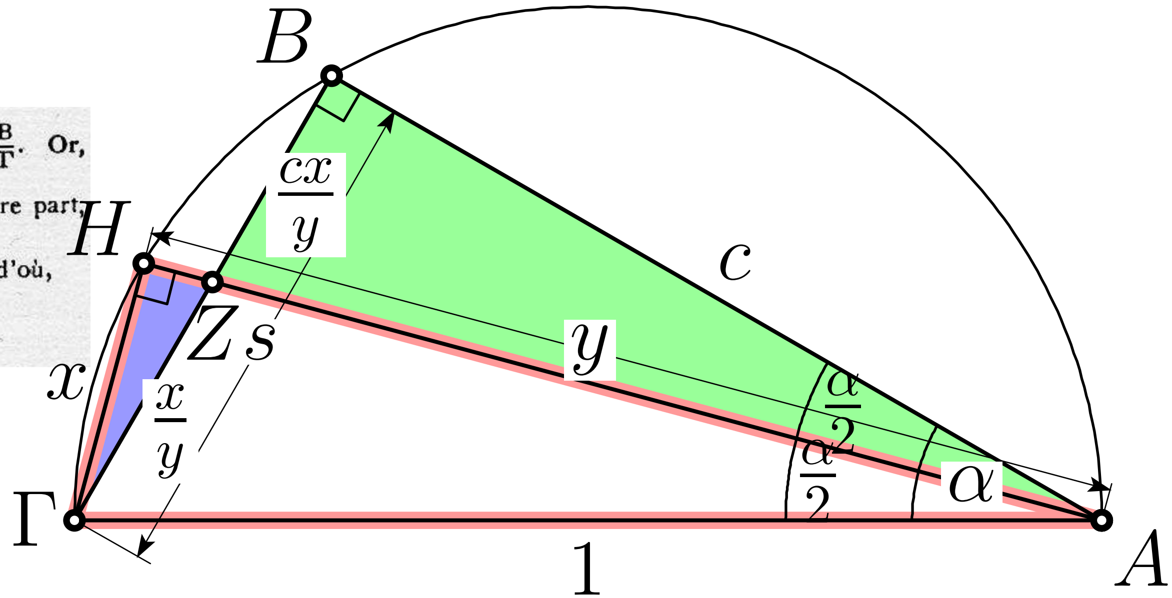
berechne $H\Gamma, \Theta\Gamma, K\Gamma, \Lambda\Gamma$

und $96 \cdot \Lambda\Gamma < \pi$.

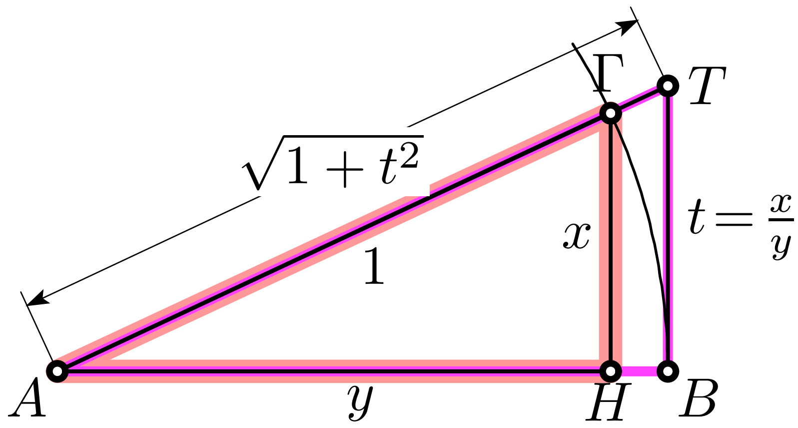
Noch etwas klarer: $(s,c)_{\text{pour } \alpha} \mapsto (x,y)_{\text{pour } \frac{\alpha}{2}}$:

d'où : $\frac{A\Gamma}{A\Gamma+AB}=\frac{\Gamma Z}{\Gamma Z+ZB}$, ou $\frac{A\Gamma}{\Gamma Z}=\frac{A\Gamma+AB}{B\Gamma}$, d'où : $\frac{AH}{H\Gamma}=\frac{A\Gamma+AB}{B\Gamma}=\frac{A\Gamma}{B\Gamma}+\frac{AB}{B\Gamma}$. Or, $\frac{A\Gamma}{B\Gamma}=\frac{1560}{780}$, et $\frac{AB}{B\Gamma}<\frac{1351}{780}$; donc $\frac{AH}{H\Gamma}<\frac{1560+1351}{780}$, ou $\frac{AH}{H\Gamma}<\frac{2911}{780}$. D'autre part, $\frac{\overline{AH}^2}{\overline{H\Gamma}^2}<\frac{(2911)^2}{(780)^2}$, d'où : $\frac{\overline{AH}^2+\overline{H\Gamma}^2}{\overline{H\Gamma}^2}<\frac{2911^2+780^2}{780^2}$, ou $\frac{\overline{A\Gamma}^2}{\overline{H\Gamma}^2}<\frac{9082321}{608400}$, d'où, comme le texte : $\frac{A\Gamma}{H\Gamma}<\frac{3013\frac{3}{4}}{780}$.

(Ver Eecke)



$$\triangle \approx \triangle \Rightarrow \Gamma Z = \frac{x}{y}, \quad \triangle \approx \triangle \Rightarrow ZB = \frac{cx}{y} \quad \Rightarrow s = \frac{x}{y}(1+c)$$



$$\frac{x}{y} = \frac{s}{c+1} = t$$

$$\sin.\tfrac{1}{2}v = \sqrt{\frac{1-\cos.v}{2}}.$$

$$\triangle \approx \triangle \Rightarrow x = \frac{t}{\sqrt{1+t^2}} = \sqrt{\frac{1-c}{2}}$$

$$H\Gamma > \frac{780}{3013\frac{3}{4}}, \quad \Theta\Gamma > \frac{240}{1838\frac{9}{11}}, \quad K\Gamma > \frac{66}{1009\frac{1}{6}}, \quad \Lambda\Gamma > \frac{66}{2017\frac{1}{4}} \quad \times 96 \Rightarrow \boxed{3\frac{1137}{8069}}$$

... und weiter ?

Zu Chongzhi (China 480 n.Chr.) mit derselben Methode:

$$3.1415926 < \pi < 3.1415927$$

Mādhava (Kerala, Indien 1350–1425 n.Chr.) ähnliche M.:

$$\pi = 3.14159265359$$

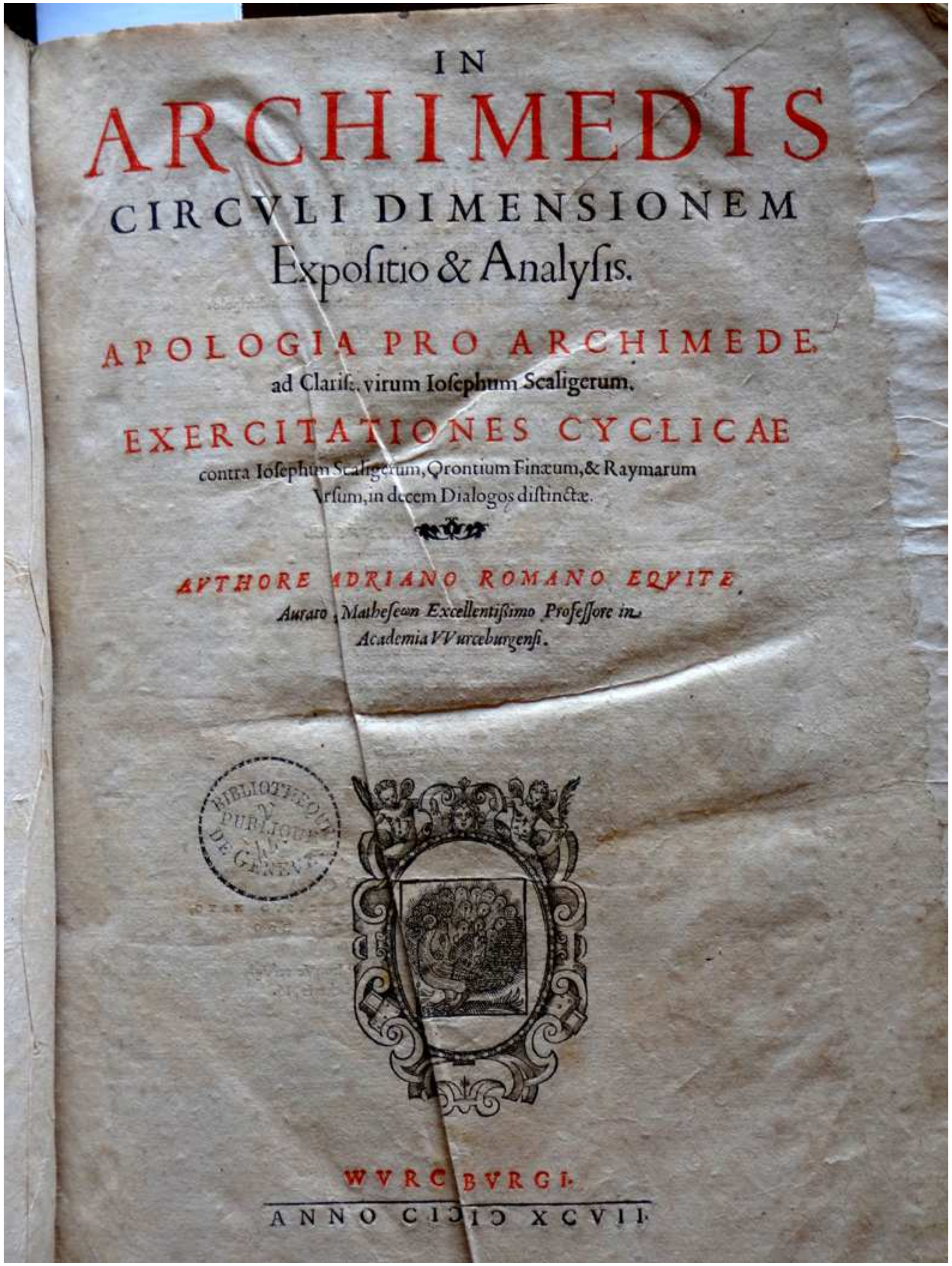
Al-Kāshī (Samarkand 1424 n.Chr.) 26× Archimedes :

$$\pi = 3.1415926535897932$$

Die Europäer waren noch etwas hinten:
Josephus Scaliger (Leyden publ. 1594):

$$\pi = \sqrt{10} = 3.162278 \quad !!!$$

Diese Behauptung des berühmten Professors Joseph Scaliger entfachte einen Sturm der Entrüstung ...


IN
ARCHIMEDIS
CIRCVLI DIMENSIONEM
Expoſitio & Analyſis.
APOLOGIA PRO ARCHIMEDE
ad Clariſs. virum Ioſephum Scaligerum.
EXERCITATIONES CYCLICAE
contra Ioſephum Scaligerum, Orontium Finæum, & Raymarum
Vrſum, in decem Dialogos diſtinctæ.
AVTHORE ADRIANO ROMANO EQVITE
Aurato, Matheſeωn Excellentiſſimo Profeſſore in
Academia VVurceburgenſi.
WVRCEBVRGI.
ANNO CIↃ IↃ XCVII.

Adriaan van Roomen
(publ. WVRCEBURGI 1597):
nein, Archimedes **ist richtig** ..
$\pi = 3.141592653589793\,2$

Sein Freund
Ludolph van Ceulen
am Ende seines Lebens (1610)
(Grabinschrift in der
Pieterskerk zu Leyden)...

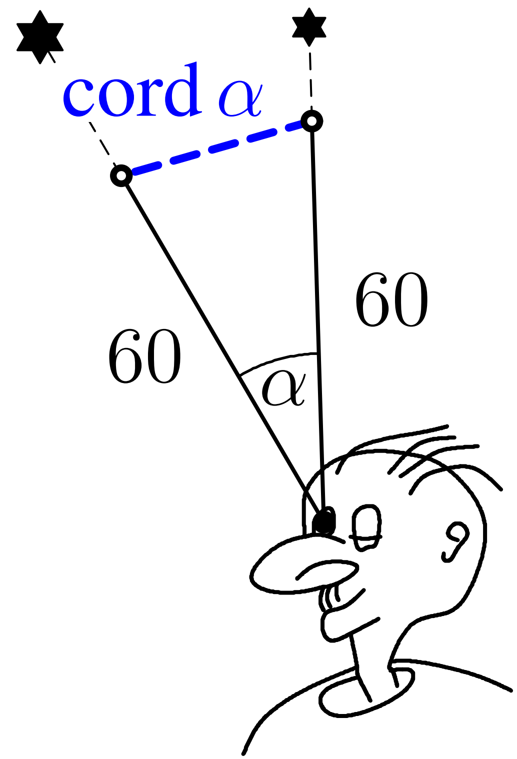
$$\pi = 3.14159265358979323846264338327950288$$

Sehnentafel des Klaudios Ptolemaios

(150 A.D., Cl. Ptolémée, Ptolemy, Tolomeo, Ptolemæus, Ptolemäus,....)

ΚΑΝΟΝΙΟΝ ΤΩΝ ΕΝ ΚΥΚΛΩ ΕΥΘΕΙΩΝ.

ΠΕΡΙΦΕΡΕΙΩΝ. Μοιρῶν.		ΕΥΘΕΙΩΝ. Μ.	Π.	Δ.	ΕΞΗΚΟΣΤΩΝ. Μ.	Π.	Δ.	Τ.
ο̄	ς″	ο̄	λα	κε	ο̄	α	β	ν
α	ο̄	α	β	ν	ο̄	α	β	ν
α	ς″	α	λδ	ιε	ο̄	α	β	ν
β	ο̄	β	ε	μ	ο̄	α	β	ν
β	ς″	β	λζ	δ	ο̄	α	β	μη
γ	ο̄	γ	η	κη	ο̄	α	β	μη
γ	ς″	γ	λθ	νβ	ο̄	α	β	μη
δ	ο̄	δ	ια	ις	ο̄	α	β	μζ
δ	ς″	δ	μβ	μ	ο̄	α	β	μζ
ε	ο̄	ε	ιδ	δ	ο̄	α	β	μς
ε	ς″	ε	με	κζ	ο̄	α	β	με
ς	ο̄	ς	ις	μθ	ο̄	α	β	μδ
ς	ς″	ς	μη	ια	ο̄	α	β	μγ
ζ	ο̄	ζ	ιθ	λγ	ο̄	α	β	μβ
ζ	ς″	ζ	ν	νδ	ο̄	α	β	μα
η	ο̄	η	κβ	ιε	ο̄	α	β	μ
η	ς″	η	νγ	λε	ο̄	α	β	λθ
θ	ο̄	θ	κδ	νδ	ο̄	α	β	λη



Wie zu lesen ?

ΚΑΝΟΝΙΟΝ ΤΩΝ ΕΝ ΚΥΚΛΩ ΕΥΘΕΙΩΝ.								
ΠΕΡΙΦΕ-ΡΕΙΩΝ.		ΕΥΘΕΙΩΝ.			ΕΞΗΚΟΣΤΩΝ.			
Μοιρῶν.		Μ.	Π.	Δ.	Μ.	Π.	Δ.	Τ.
ō	ς″	ō	λα	κε	ō	α	β	ν
α	ō	α	β	ν	ō	α	β	ν
α	ς″	α	λδ	ιε	ō	α	β	ν
β	ō	β	ε	μ	ō	α	β	ν
β	ς″	β	λζ	δ	ō	α	β	μη
γ	ō	γ	η	κη	ō	α	β	μη
γ	ς″	γ	λθ	νβ	ō	α	β	μη
δ	ō	δ	ια	ις	ō	α	β	μζ
δ	ς″	δ	μβ	μ	ō	α	β	μζ
ε	ō	ε	ιδ	δ	ō	α	β	μς
ε	ς″	ε	με	κζ	ō	α	β	με
ς	ō	ς	ις	μθ	ō	α	β	μδ
ς	ς″	ς	μη	ια	ō	α	β	μγ
ζ	ō	ζ	ιθ	λγ	ō	α	β	μβ
ζ	ς″	ζ	ν	νδ	ō	α	β	μα
η	ō	η	κβ	ιε	ō	α	β	μ
η	ς″	η	νγ	λε	ō	α	β	λθ
θ	ō	θ	κδ	νδ	ō	α	β	λη

gr.		gr.		rm.	
α	1	A		A	
β	2	B		B	
γ	3	Γ		C	↓
δ	4	Δ		D	
ε	5	E		E	
(Ϝ)	6	F		F	
ζ	7	Z	↗		G
η	8	H		H	
ϑ	9	Θ	↗		I
ι	10	I		I	J
κ	20	K		K	
λ	30	Λ		L	
μ	40	M		M	Z
ν	50	N		N	↓

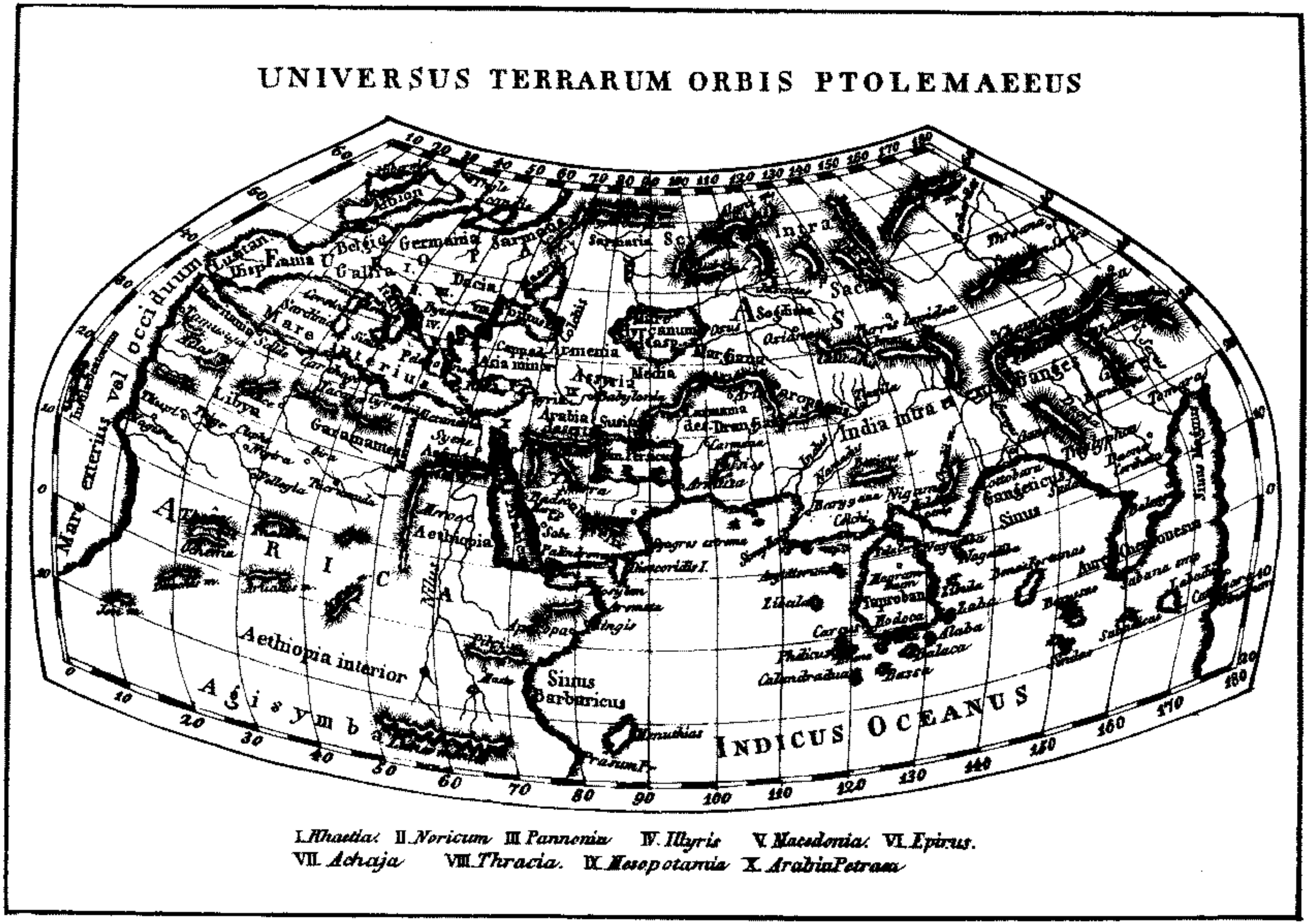
zur Kontrolle:

ΚΑΝΟΝΙΟΝ ΤΩΝ ΕΝ ΚΥΚΛΩ ΕΥΘΕΙΩΝ.

ΠΕΡΙΦΕΡΕΙΩΝ.		ΕΥΘΕΙΩΝ.			ΕΞΗΚΟΣΤΩΝ.			
Μοιρῶν.		Μ.	Π.	Δ.	Μ.	Π.	Δ.	Τ.
ō	ς''	ō	λα	κε	ō	α	β	ν
α	ō	α	β	ν	ō	α	β	ν
α	ς''	α	λδ	ιε	ō	α	β	ν
β	ō	β	ε	μ	ō	α	β	ν
β	ς''	β	λζ	δ	ō	α	β	μη
γ	ō	γ	η	κη	ō	α	β	μη
γ	ς''	γ	λθ	νβ	ō	α	β	μη
δ	ō	δ	ια	ις	ō	α	β	μζ
δ	ς''	δ	μβ	μ	ō	α	β	μζ
ε	ō	ε	ιδ	δ	ō	α	β	μς
ε	ς''	ε	με	κζ	ō	α	β	με
ς	ō	ς	ις	μθ	ō	α	β	μδ
ς	ς''	ς	μη	ια	ō	α	β	μγ
ζ	ō	ζ	ιθ	λγ	ō	α	β	μβ
ζ	ς''	ζ	ν	νδ	ō	α	β	μα
η	ō	η	κβ	ιε	ō	α	β	μ
η	ς''	η	νγ	λε	ō	α	β	λθ
θ	ō	θ	κδ	νδ	ō	α	β	λη

0	30	0	31	25
1	0	1	2	50
1	30	1	34	15
2	0	2	5	39
2	30	2	37	4
3	0	3	8	28
3	30	3	39	53
4	0	4	11	17
4	30	4	42	40
5	0	5	14	4
5	30	5	45	27
6	0	6	16	49
6	30	6	48	11
7	0	7	19	33
7	30	7	50	54
8	0	8	22	15
8	30	8	53	35
9	0	9	24	54

Ptolemaios' Geographie (Γῆ̂ . . . γράφω . . . μετρέω):



Ptolemaios vermisst 8000 Städte, z.B. ...

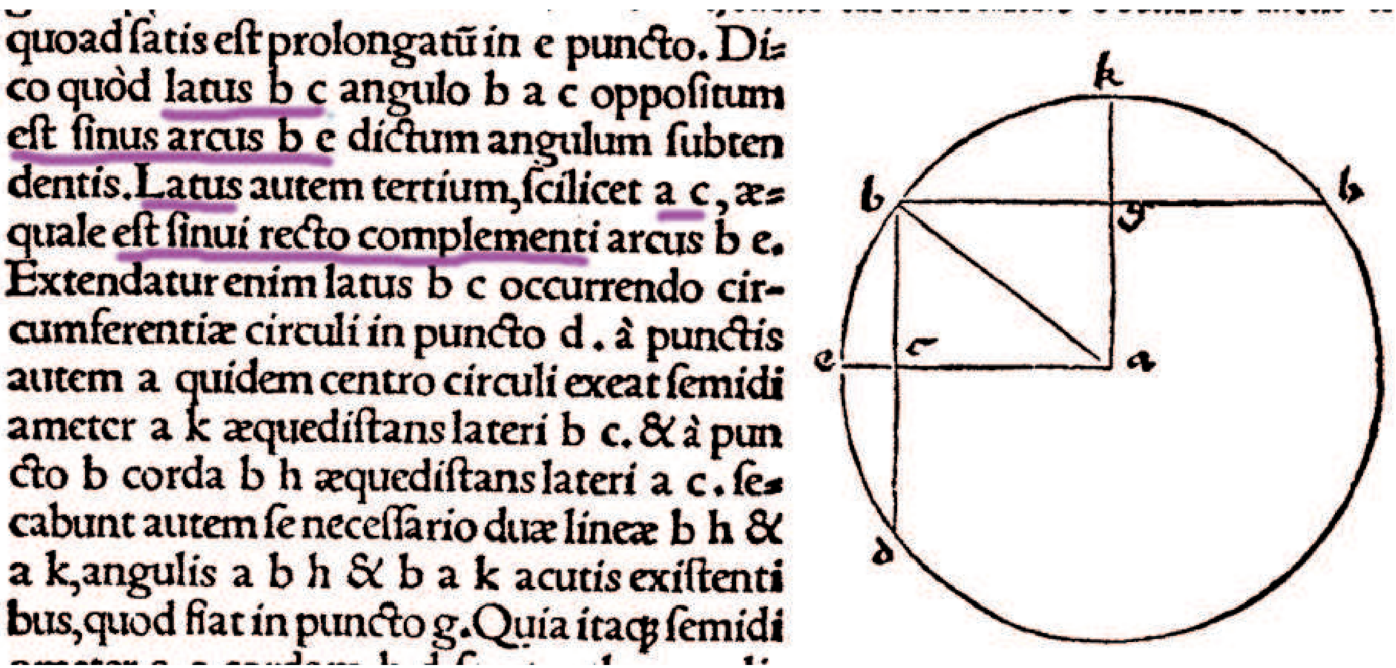
ὑφ' οὓς Παρίσιοι, καὶ πόλις
Παρισίων Λουκοτεκία . . . . κγ ∠ μη ∠

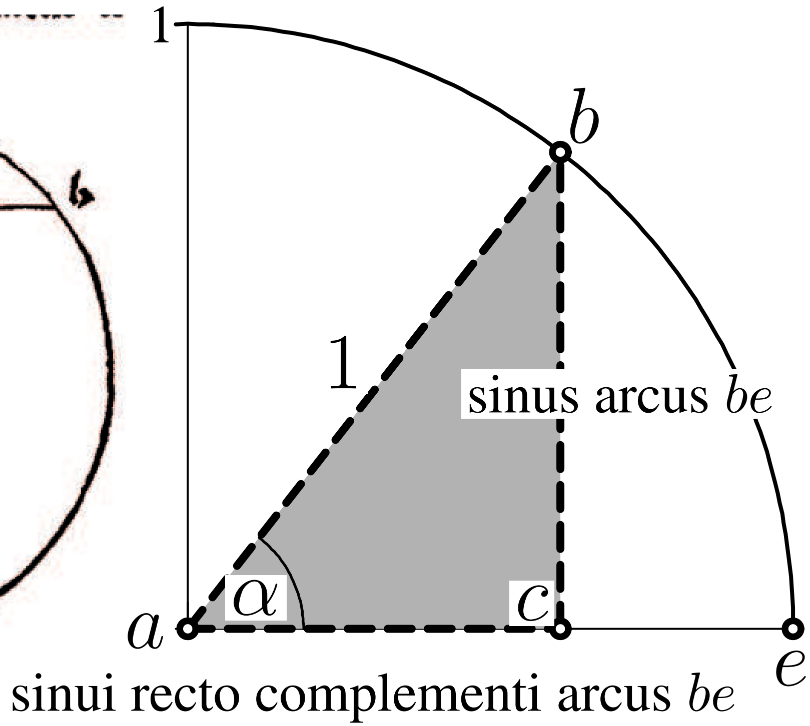
Längengrade fehlerhaft bei 180° ⇒ Chr. Columbus !!

Johannes Regiomontanus (1436–1476):

mit 11 Uni Leipzig, mit 12 erstes Buch, mit 14 Uni Wien, mit 25 Rom
liest Archimedes, Ptolemeus im Original, sowie arabische Schriften ⇒

De triangulis omnimodis libri quinque, (publ. 1533);

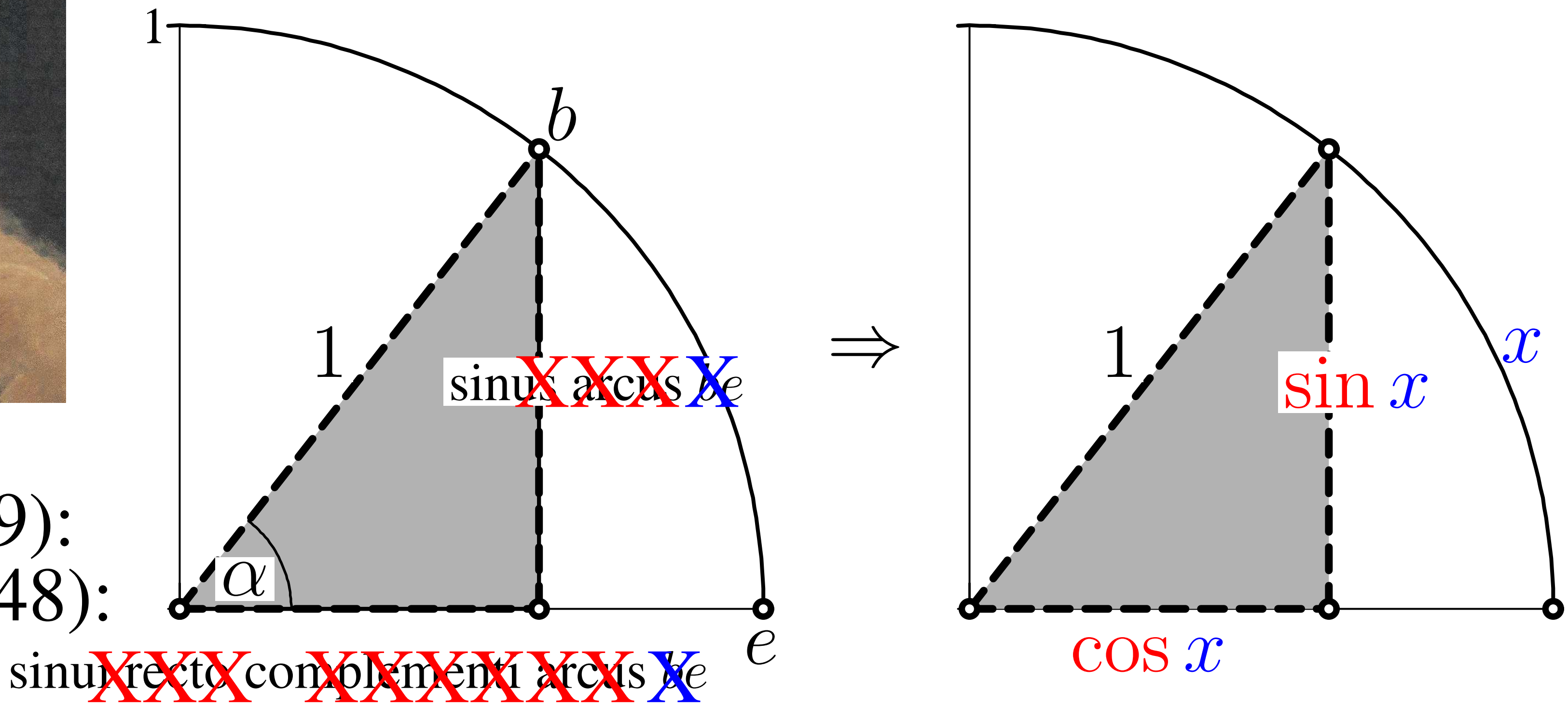

quoad ſatis eſt prolongatũ in e puncto. Di=
co quòd latus b c angulo b a c oppoſitum
eſt ſinus arcus b e dictum angulum ſubten
dentis. Latus autem tertium, ſcilicet a c, æ=
quale eſt ſinui recto complementi arcus b e.
Extendatur enim latus b c occurrendo cir-
cumferentiæ circuli in puncto d. à punctis
autem a quidem centro circuli exeat ſemidi
ameter a k æquediſtans lateri b c. & à pun
cto b corda b h æquediſtans lateri a c. ſe=
cabunt autem ſe neceſſario duæ lineæ b h &
a k, angulis a b h & b a k acutis exiſtenti
bus, quod fiat in puncto g. Quia itaq; ſemidi



Compositio tabularum sinuum (publ. 1541)

(“SEQVITVR NVNC EIVSDEM IOANNIS Regiomontani tabula sinuum,
per singula minuta extensa . . .”).

L. Euler
E14 (1729):
E101 (1748):

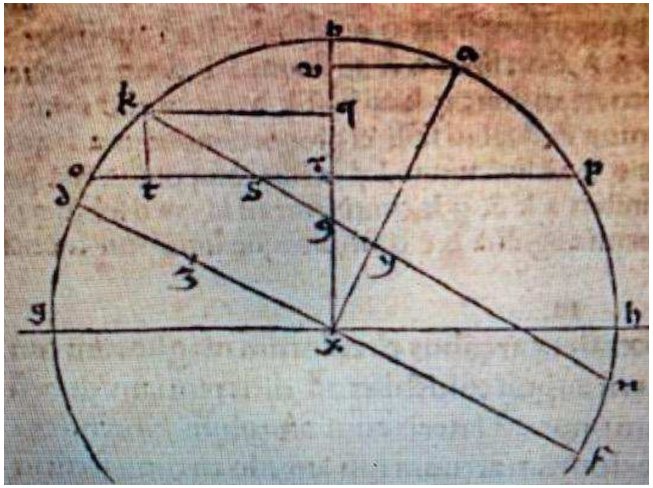


Für die Additionstheoreme (*Introductio* Cap. VIII, §128)

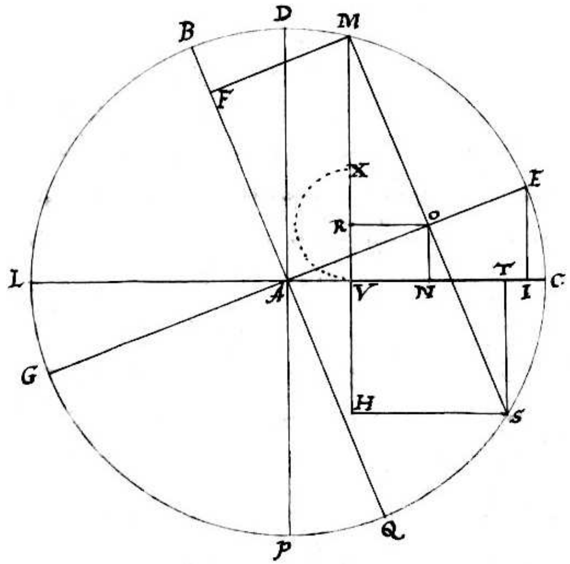
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

schreibt er einfach “Hinc vero etiam constat ...”
sie waren also schon vor 3 Jhten. jedem Anfänger bekannt...

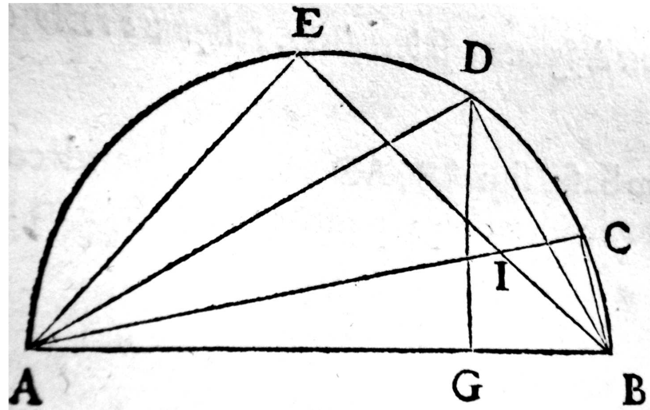
Historische Beweise ??



(Regiomontanus 1533, p. 128)

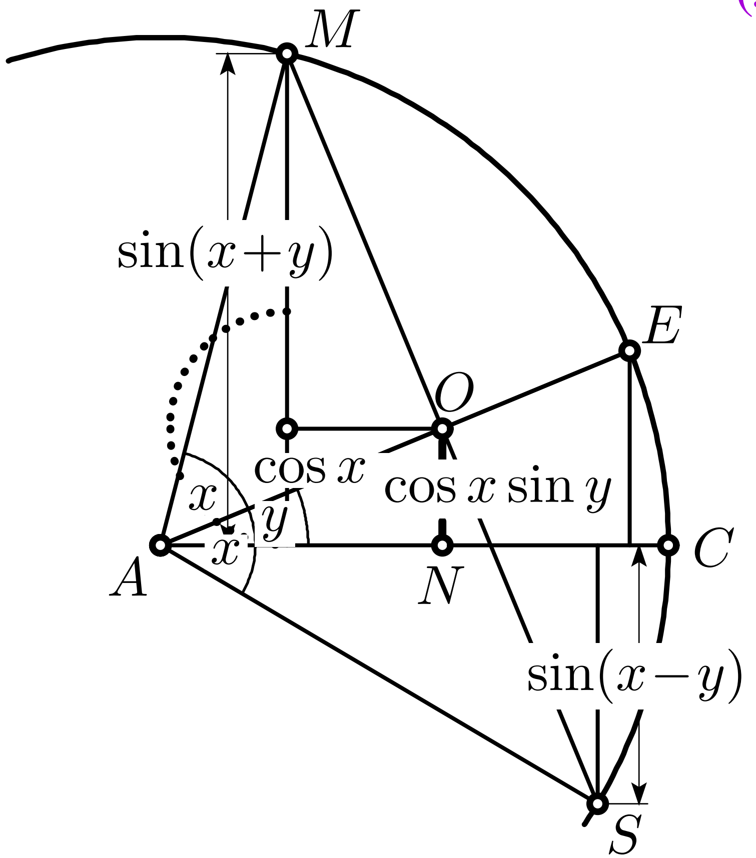


(Jost Bürgi, 1584)

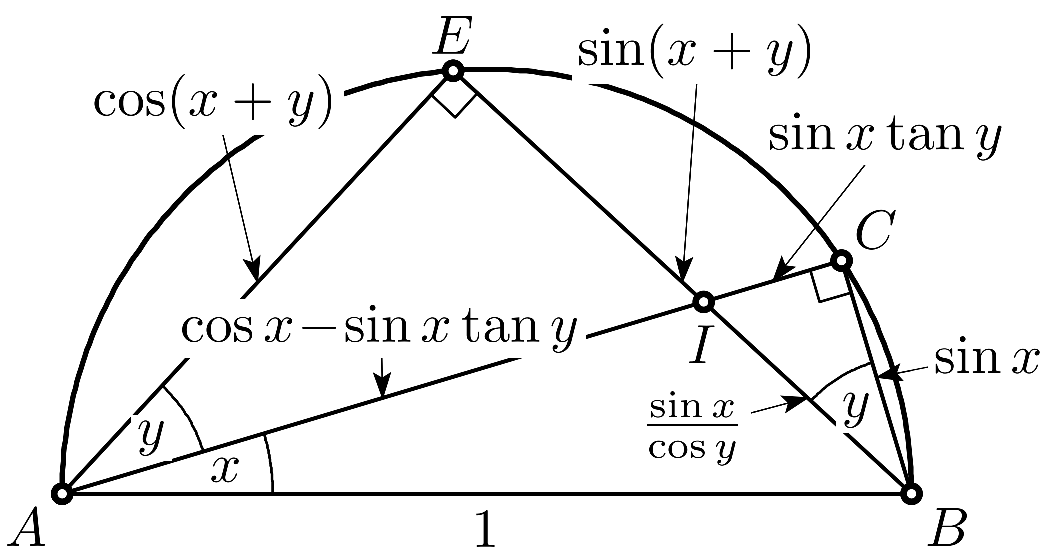


(F. Viète, 1593, impr. 1646)

(vide Archimedes !!)



$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$$



$$AE = AI\cos y = \cos x \cos y - \sin x \sin y$$

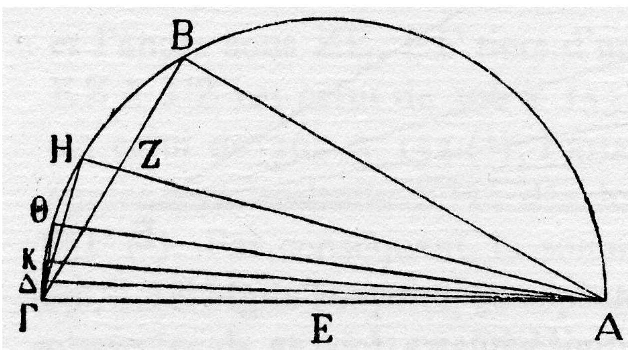
$$EI = AI\sin y \Rightarrow EI + IB = \sin(x+y)$$

Historische Berechnungen der Sinustafeln

Das erste Problem war die Berechnung von

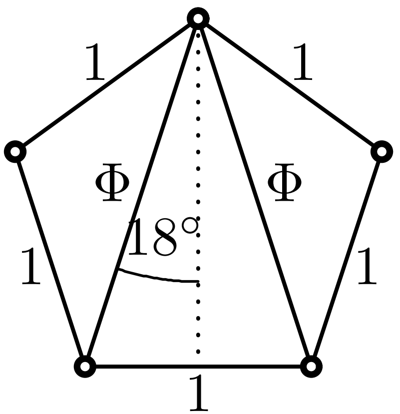
$$\sin 1^\circ = 0.017452406437283\,51 = 1\ 2\ 49\ 43\ 11\ 14\ 44\ 16..$$

alle anderen Sinuswerte durch Additionstheorem.
Bereits Archimedes' Resultate für $B\Gamma$, $H\Gamma$, $\Theta\Gamma$, $K\Gamma$, ... sind



n	α	$\sin\alpha$	$\sin\alpha/\alpha$	Basis 60			
6	30.0000	0.50000000	0.01666667	1	0	0	0
12	15.0000	0.25881905	0.01725460	1	2	6	59
24	7.5000	0.13052619	0.01740349	1	2	39	9
48	3.7500	0.06540313	0.01744083	1	2	47	13
96	1.8750	0.03271908	0.01745018	1	2	49	14
192	0.9375	0.01636173	0.01745251	1	2	49	44

Sinuswerte für immer kleinere α, aber nie $\alpha = 1^\circ$.
Lösung: $\sin 1 \approx \sin\alpha/\alpha$ (bereits genügend für Ptolemäus).



Bemerkung.

$\sin 18^\circ = 1/(2\Phi)$ (gold. Schnitt)

$\Rightarrow \sin 3^\circ = \sin(18^\circ - 15^\circ)$ (exakt bekannt)

$x = \sin 1^\circ \Rightarrow -4x^3 + 3x = \sin 3^\circ$ (Al-Kāshī 1429)

Parameśvara (Indien, ca. 1380–1460) (Dank an Ph. Henry für Zitat)

Ist einmal π bekannt (siehe **Mādhava** oben), gibt es einfache Methode für $\sin\alpha$, α beliebig, beschrieben in Form von 37 Sanskrit Verszeilen, beginnend mit

> cāpāj jīvāpi sādhyā syād aviśeṣākhyakarmaṇā
> ardhīkuryād iṣṭacāpaṃ tadardham ca tathā punaḥ

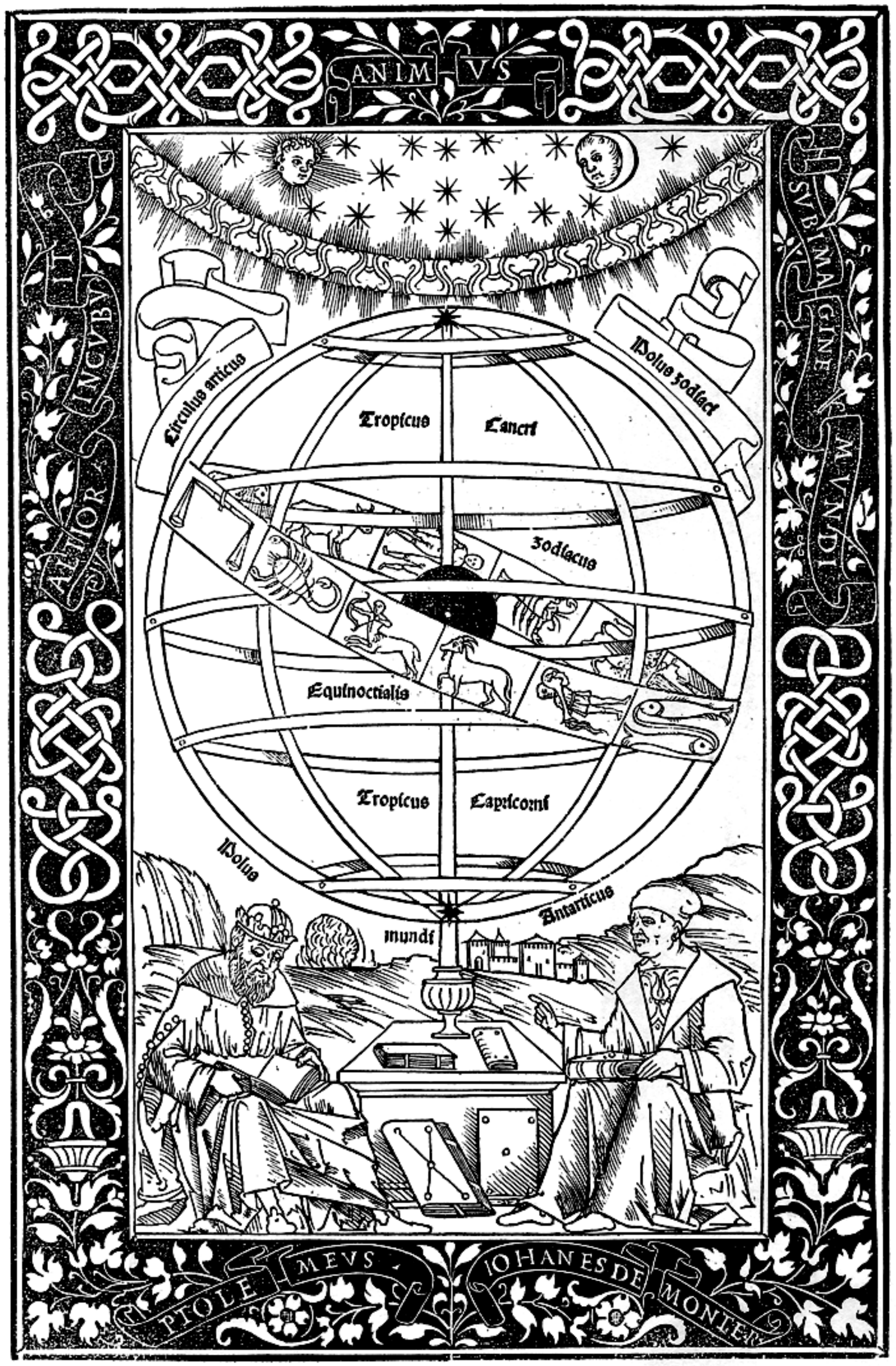
die wir aber nicht im Original weiterlesen, auch nicht in der engl. Übersetzung von Kim Plofker (1996), denn aus 37 Verszeilen machen wir 10 Fortranzeilen:

```
PI=3.14159265358979324D0
S=PI/(1024.D0*180.D0)
C=SQRT(1.D0-S**2)
WRITE(6,*)0,S,C
DO I=1,10
   STORE=2.D0*S*C
   C=1.D0-2.D0*S**2
   S=STORE
   WRITE(6,*)I,S,C
END DO
```

0	0.00001704423098	0.99999999985475
1	0.00003408846195	0.99999999941899
2	0.00006817692386	0.99999999767595
3	0.00013635384740	0.99999999070381
4	0.00027270769226	0.99999996281526
5	0.00054541536423	0.99999985126103
6	0.00109083056622	0.99999940504416
7	0.00218165983444	0.99999762017735
8	0.00436330928496	0.99999048072073
9	0.00872653549880	0.99996192306417
10	0.01745240643813	0.99984769515638

Die **Idee** ist, mit einem beliebigen α (wir nehmen, what else ? $\alpha = 1^\circ$) dieses α durch 2^n zu dividieren, mit z.B. $n = 10$, dann ist $\sin(2^{-n}\alpha) \approx \mathrm{arc}(2^{-n}\alpha)$, $\cos = \sqrt{1 - \sin^2}$, und wir gehen in n Schritten mittels $\sin 2x = 2\sin x\cos x$, $\cos 2x = 1 - 2\sin^2 x$ nach oben.

Ptolemäische Kosmologie



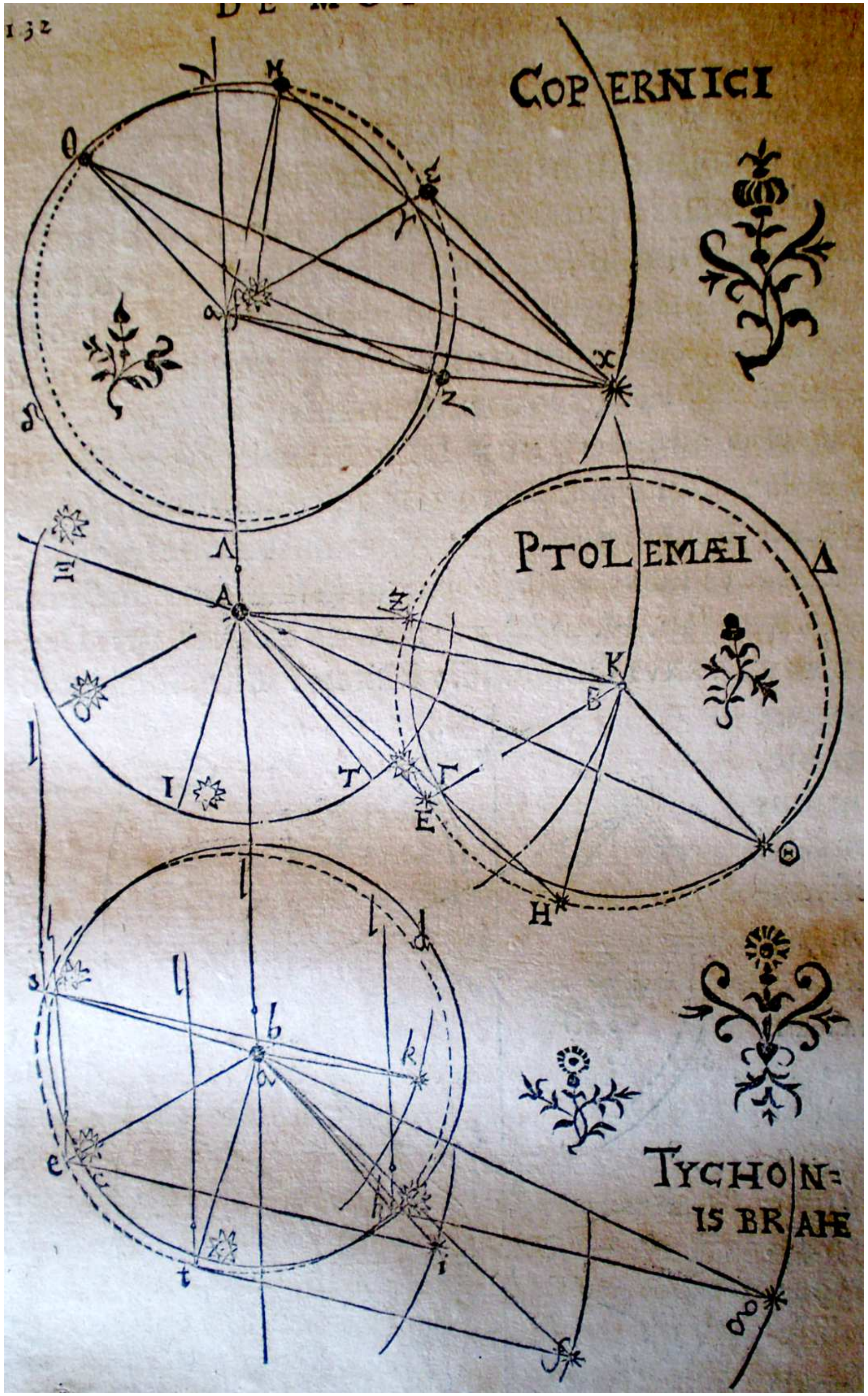
(**Almagest** de Ptolemy, pub.1496)

(Copernicus, publ. 1543)

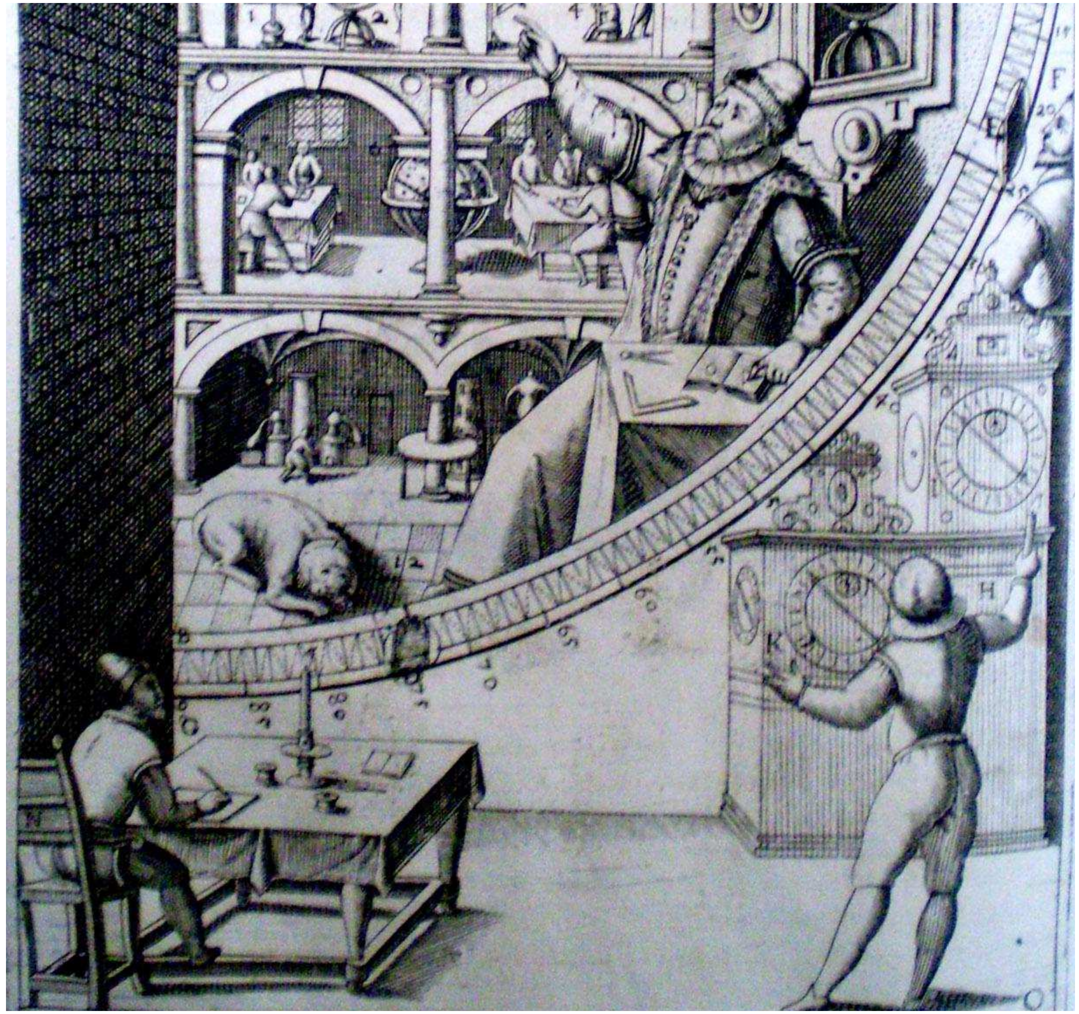
Tycho Brahe (1546 – 1601) **Joh. Kepler** (1571 – 1630)

(in Prag seit 1599)

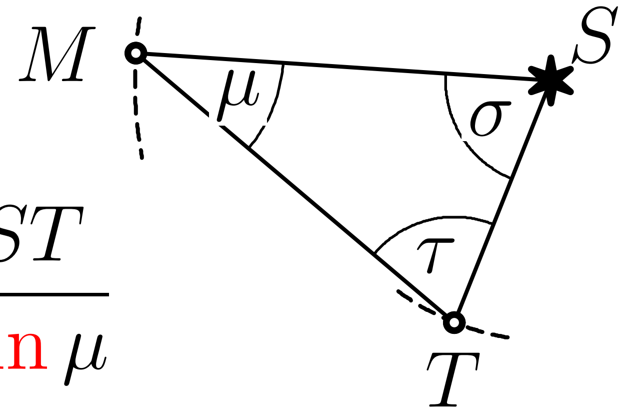
(in Prag seit 1600)



(Abb. von Kepler 1609)

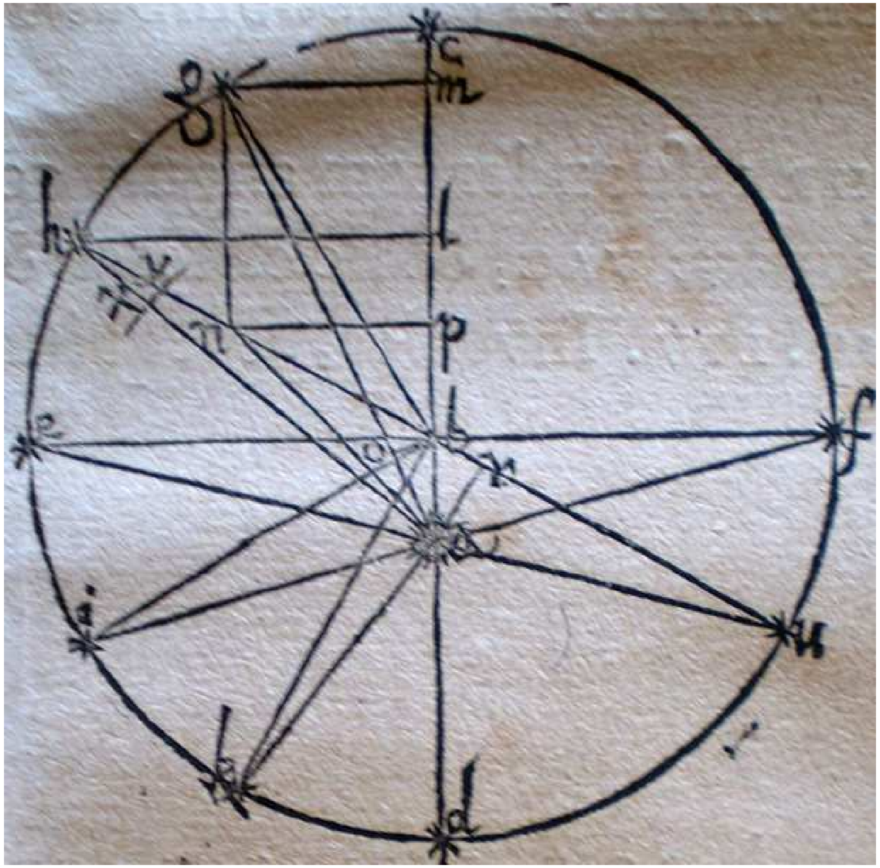


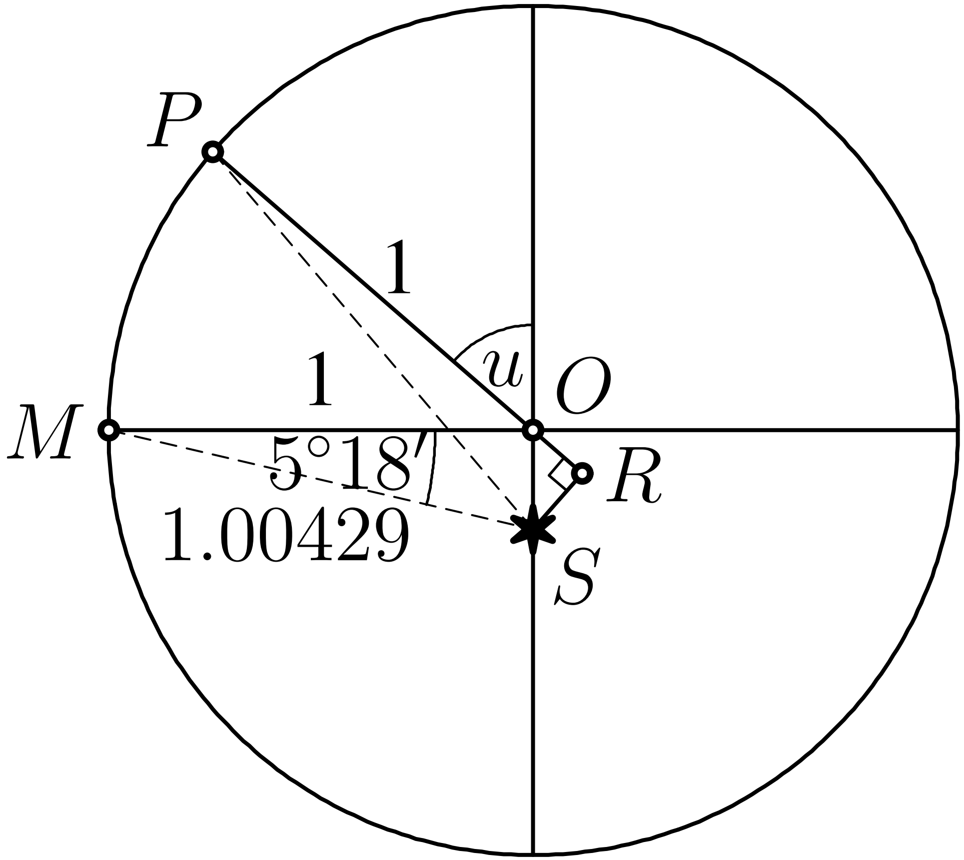
viele extrem genaue Abst. SM



$$\frac{SM}{\sin\tau} = \frac{ST}{\sin\mu}$$

Problem: Bei **Mars** ist für exzentrisches Kreismodell Abst. SM bei Opticæ maximæ um Faktor 1.00429 zu gross !!

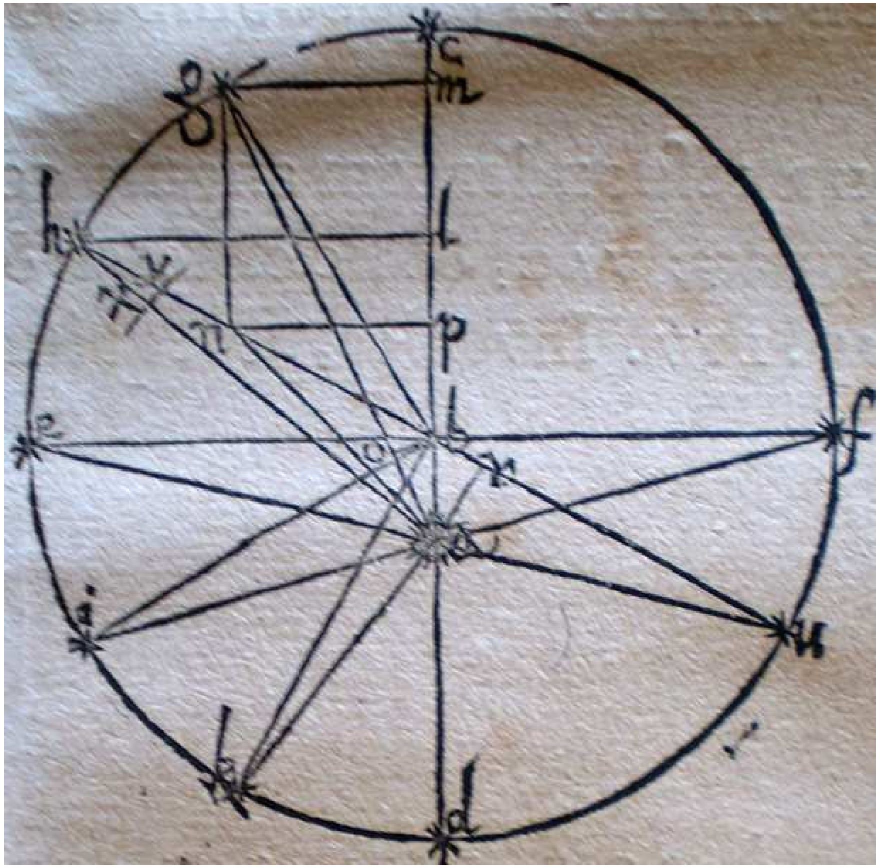


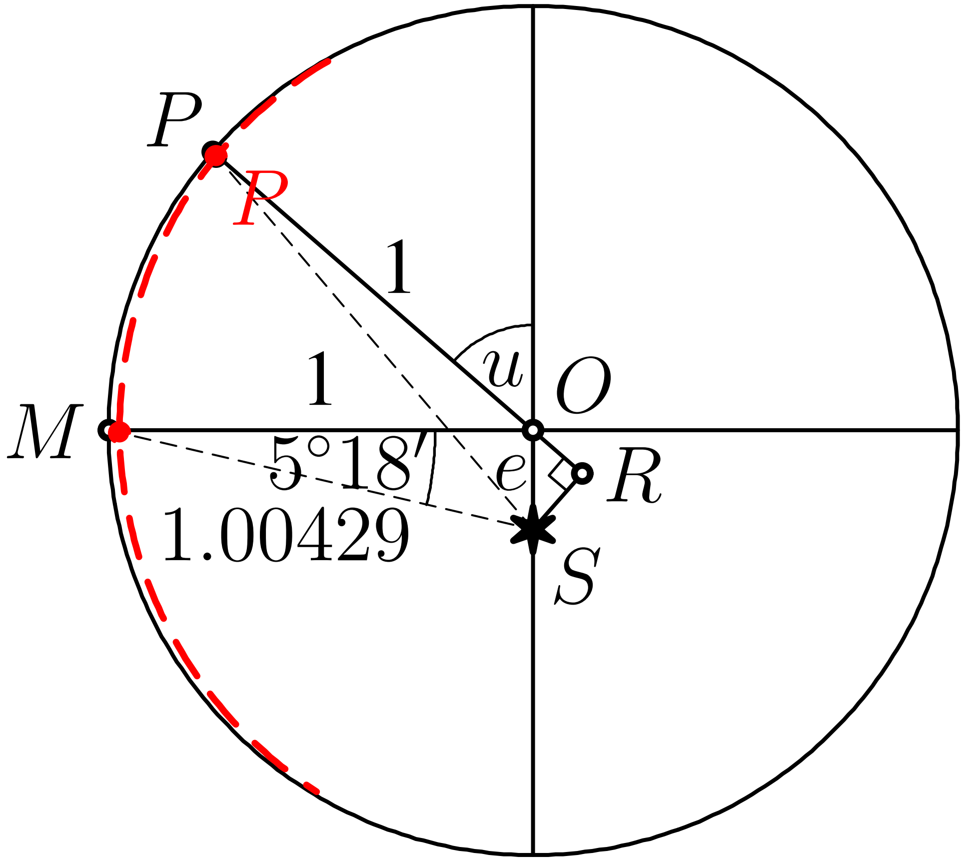


Kepler: $1.00429 = 1/\cos 5°18'$

plane nihil dictum eſſe, itaque futilem fuiſſe meum de Marte triumphum; forte fortuito incido in ſecantem anguli 5.18′. quæ eſt menſura æquationis Opticæ maximæ. Quem cum viderem eſſe 100429, hic quaſi e ſomno expergefactus, & novam lucem intuitus, ſic cœpi ratio-

Kepler "somno expergefactus, & novam lucem intuitus":





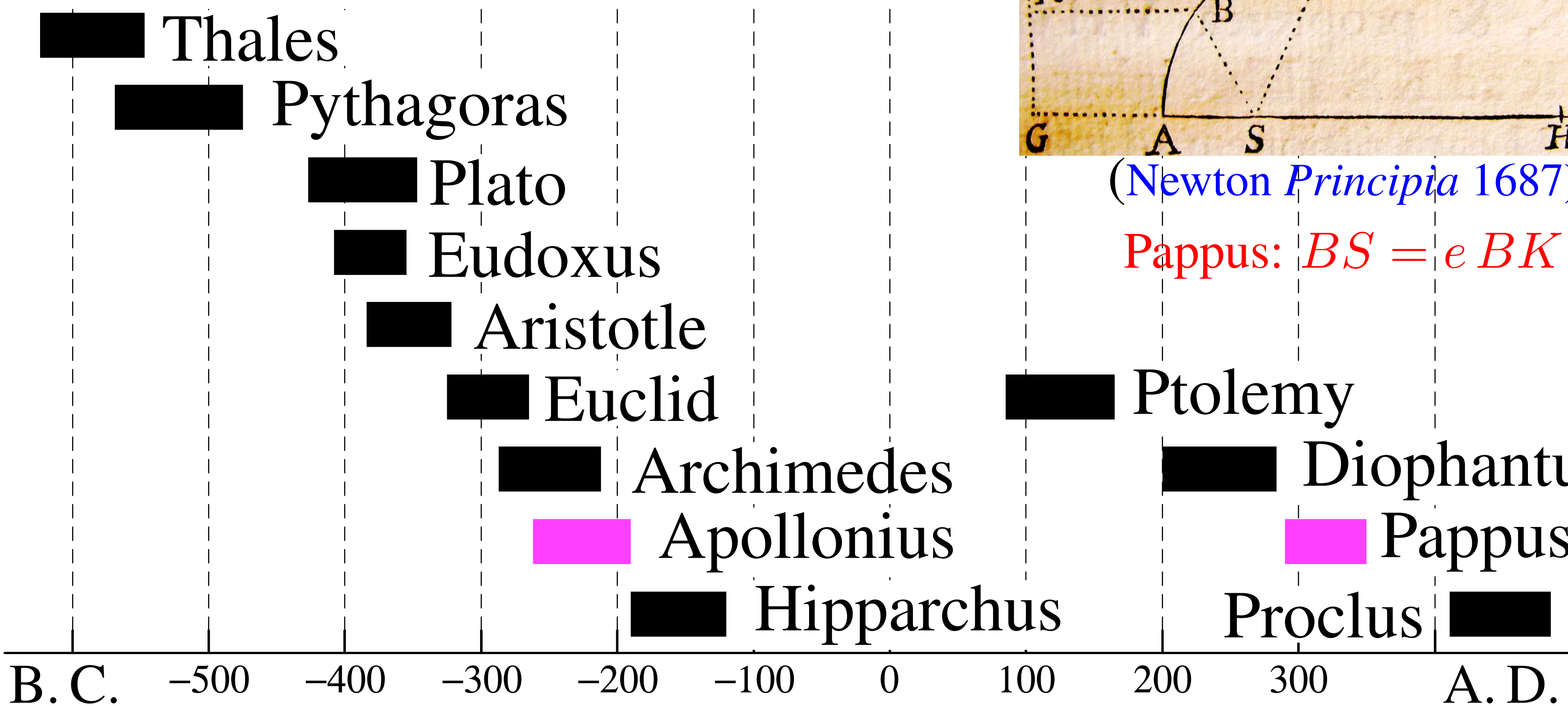
Idee: **ersetze Hypotenuse durch Kathete** $P \mapsto P$

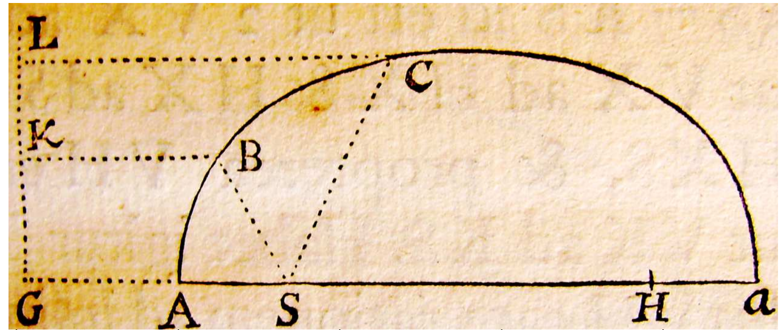
$$PS = PR = 1 + e \cos u$$

Eigentlich absurde Idee, **ABER:**

"Diese Abstände werden bestätigt durch sehr zahlreiche und sehr genaue Messungen" (Ende von Kap. 56)

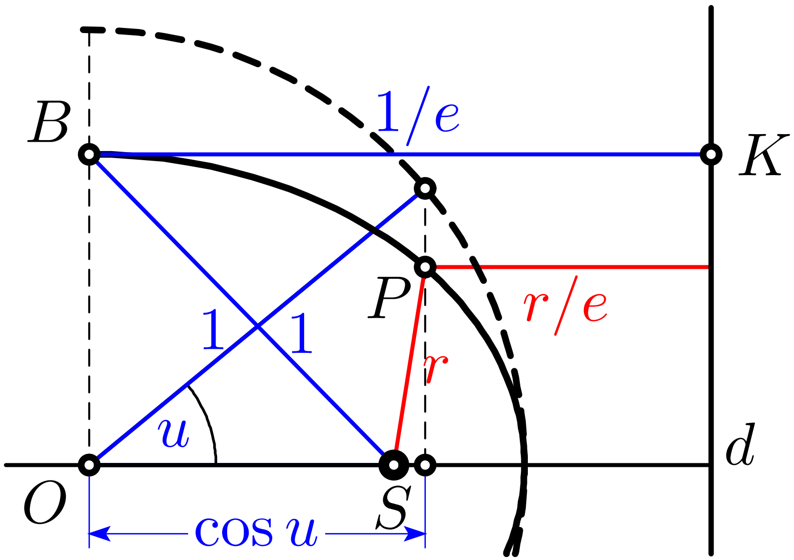
Noch zwei Griechen ...





(Newton *Principia* 1687)

Pappus: $BS = e\,BK$

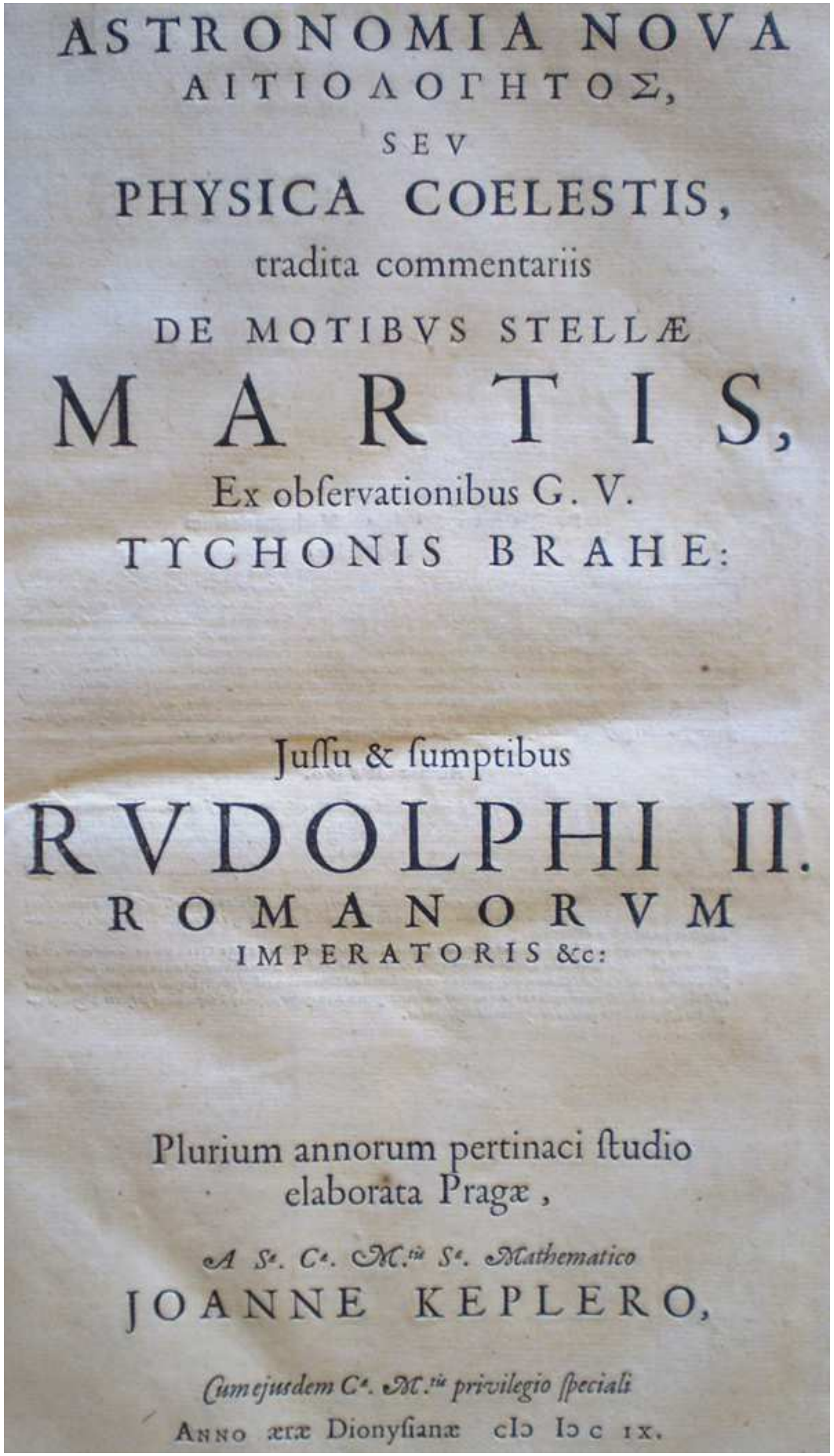


$$r/e + \cos u = 1/e$$

$$r = 1 \pm e\cos u$$

$\Rightarrow$ **Bahn ist Ellipse !!**

Johannes Kepler "de Marte triumphum" (1609):



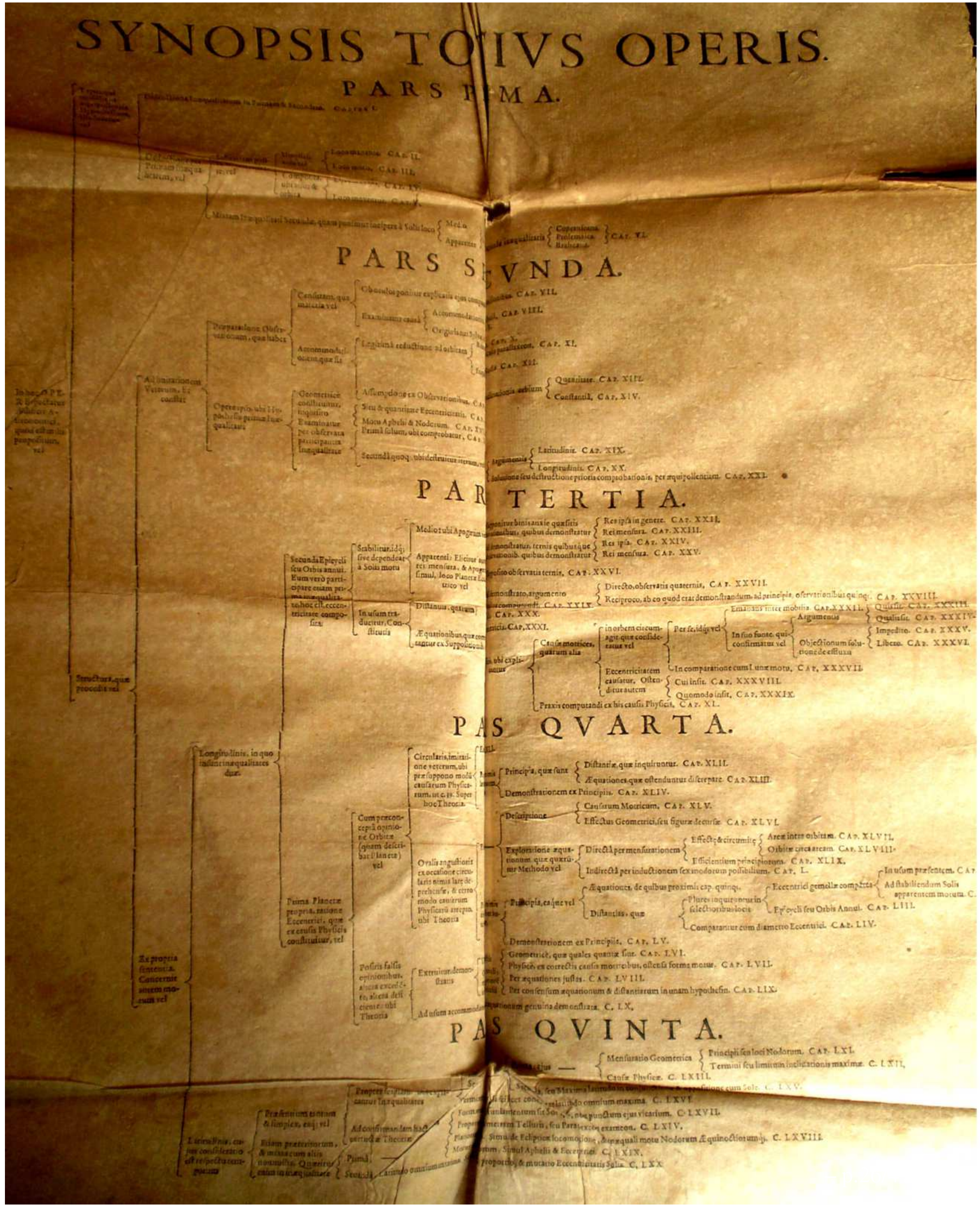
ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:

Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

Plurium annorum pertinaci ſtudio
elaborata Pragæ,

A S.ᵃ C.ᵃ M.ᵗⁱˢ S.ᵒ Mathematico
JOANNE KEPLERO,

Cum ejuſdem C.ᵃ M.ᵗⁱˢ privilegio ſpeciali
ANNO æræ Dionyſianæ cIↃ IↃ c IX.



SYNOPSIS TOTIVS OPERIS.

PARS PRIMA.

PARS SECVNDA.

PARS TERTIA.

PARS QVARTA.

PARS QVINTA.

Zoom auf PARS TERTIA et QVARTA

PARS TERTIA.

Res ipsa in genere. CAP. XXII.
Rei mensura. CAP. XXIII.
Res ipsa. CAP. XXIV.
Rei mensura. CAP. XXV.
Proposito observatis ternis. CAP. XXVI.
Directo, observatis quaternis. CAP. XXVII.
Reciproco, ab eo quod erat demonstrandum, ad principia, observationibus quinque. CAP. XXVIII.
Causæ motrices, quarum alia
Emanans inter mobilia. CAP. XXXII.
Quis sit. CAP. XXXIII.
Qualis sit. CAP. XXXIV.
Impedito. CAP. XXXV.
Libero. CAP. XXXVI.
Objectionum solutione de effluxu
In comparatione cum Lunæ motu. CAP. XXXVII.
Eccentricitatem causatur. Ostenditur autem
Cui insit. CAP. XXXVIII.
Quomodo insit. CAP. XXXIX.
Praxis computandi ex his causis Physicis. CAP. XL. ← Kap. 40: Flächensatz

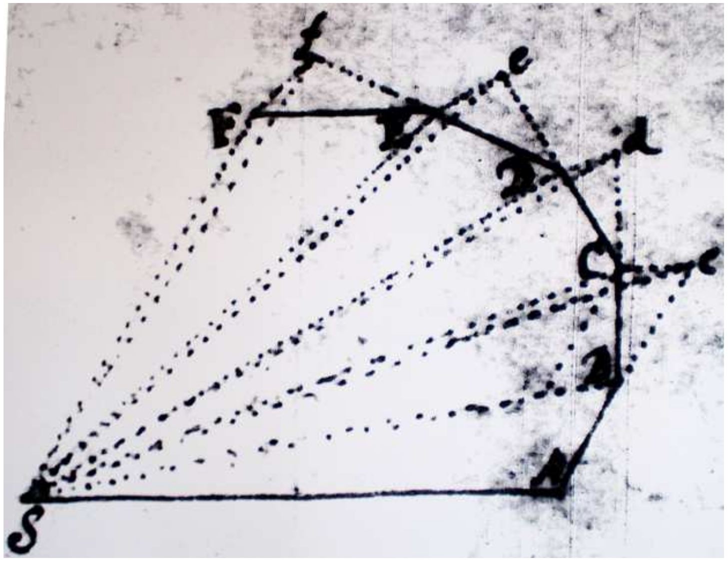
PARS QVARTA.

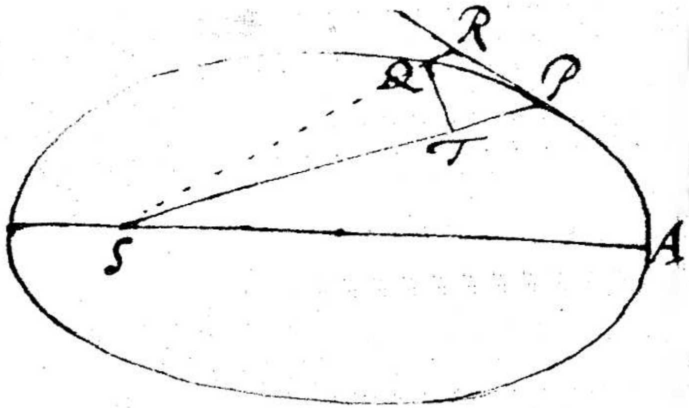
Principia, quæ sunt
Distantiæ, quæ inquiruntur. CAP. XLII.
Æquationes, quæ ostenduntur discrepare. CAP. XLIII.
Demonstrationem ex Principiis. CAP. XLIV.
Descriptione
Causarum Motricum. CAP. XLV.
Effectus Geometrici, seu figuræ decursæ. CAP. XLVI.
Exploratione æquationum quæ quæruntur Methodo vel
Directa per mensurationem
Areæ intra orbitam. CAP. XLVII.
Orbitæ circa aream. CAP. XLVIII.
Efficientium principiorum. CAP. XLIX.
Indirecta per inductionem sex modorum possibilium. CAP. L.
Principia, eaque vel
Æquationes, de quibus proximis cap. quinque.
Distantias, quæ
Comparantur cum diametro Eccentrici. CAP. LIV.
Demonstrationem ex Principiis. CAP. LV.
Geometricè, quæ quales quantæ sint. CAP. LVI. ← Kap. 56: Formel der Ellipse
Physicè, ex correctis causis motricibus, ostensa forma motus. CAP. LVII.
Per æquationes justas. CAP. LVIII.
Per consensum æquationum & distantiarum in unam hypothesin. CAP. LIX.
Æquationum genuina demonstrata. C. LX.

Newton: Allg. Gravitationsgesetz (1684,1687)

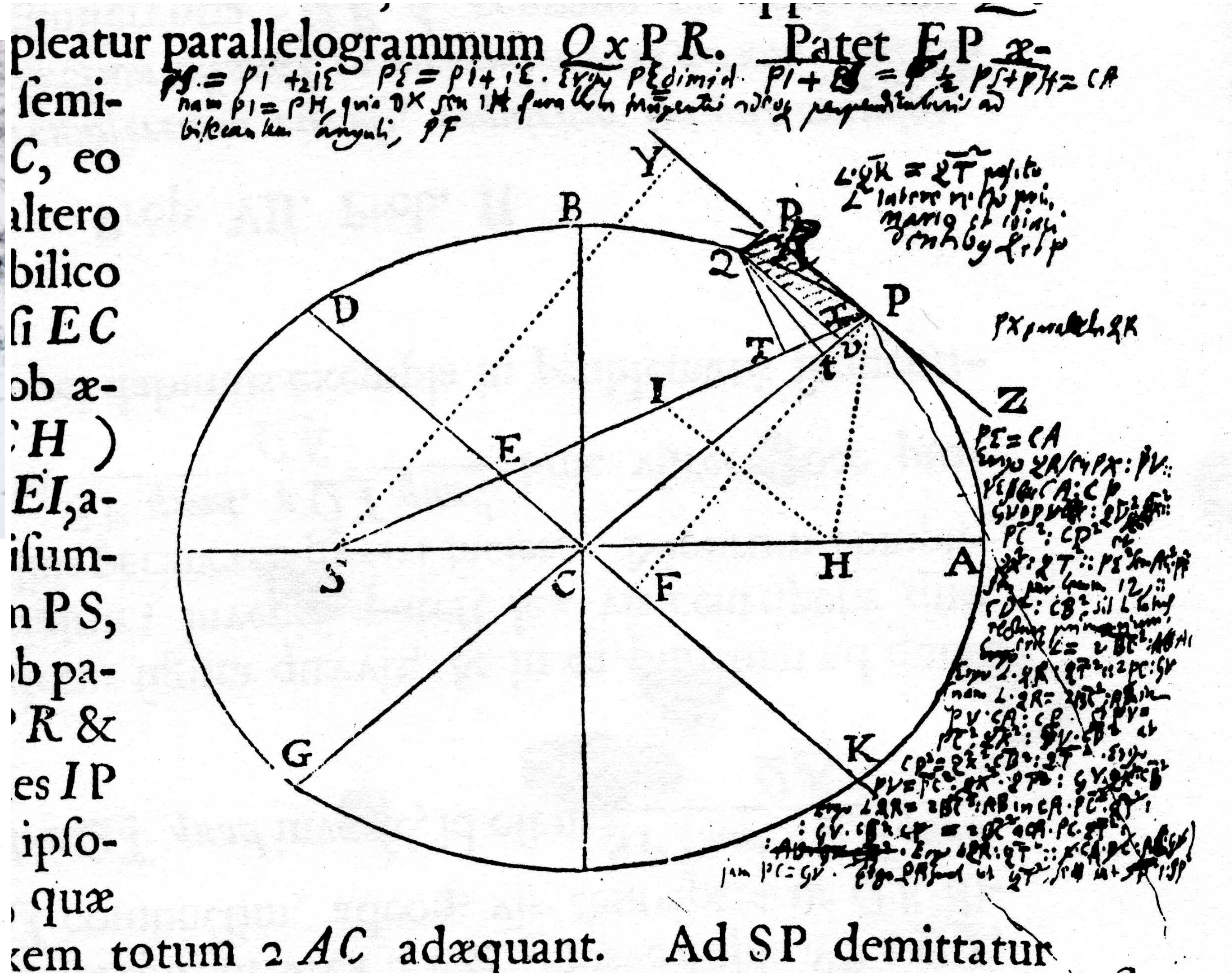
‘The Principia is .. the greatest scientific book ever written.’

(http://mathshistory.st-andrews.ac.uk/Biographies/Newton.html)





(from ms. Add. 3965^{7a})



(Newton’s Princ. 1687, Leibniz’ handwriting)

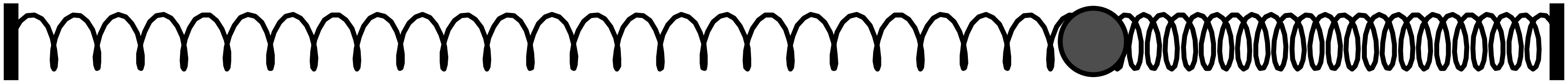
Theorem. Kraft prop. RQ prop. QT^2 prop. $1/r^2$.

Zurück zur Erde: ein Problem der Mechanik:

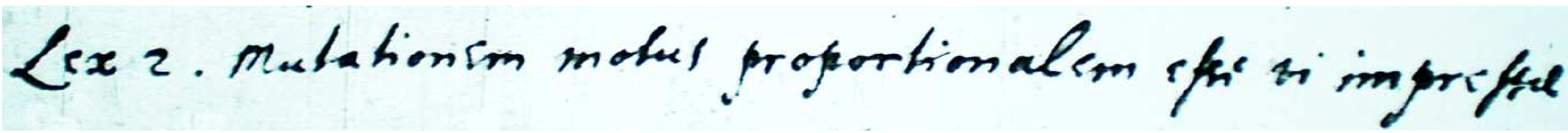
Voilà les fondements de la physique moderne, céleste et terrestre.

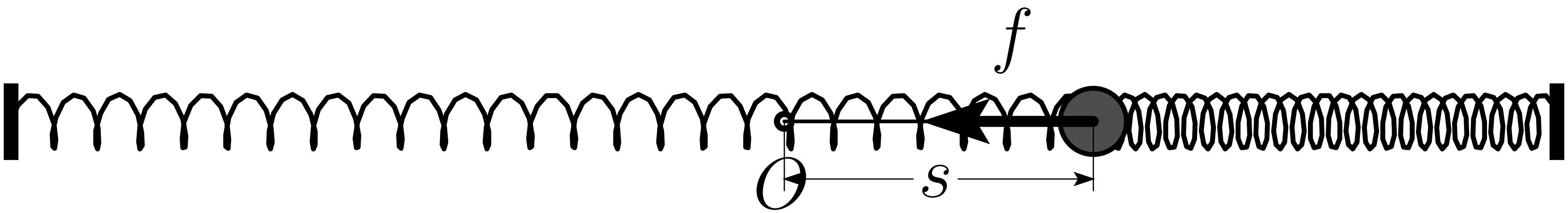
(Delambre 1821; see M. Caspar, *J. Kepler, Werke 3*, p. 256)

Wie bewegt sich ein an Federn angehängtes Gewicht ?



Bewegung verursacht durch Kraft:


Lex 2. Mutationem motus proportionalem esse vi impressae

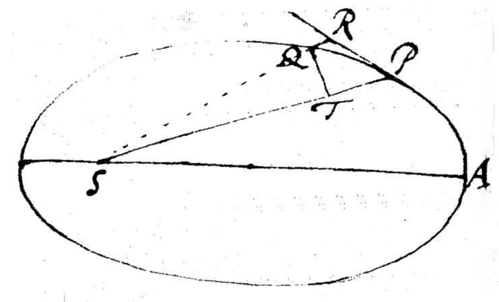


Hookesches Gesetz (Robert Hooke 1676):

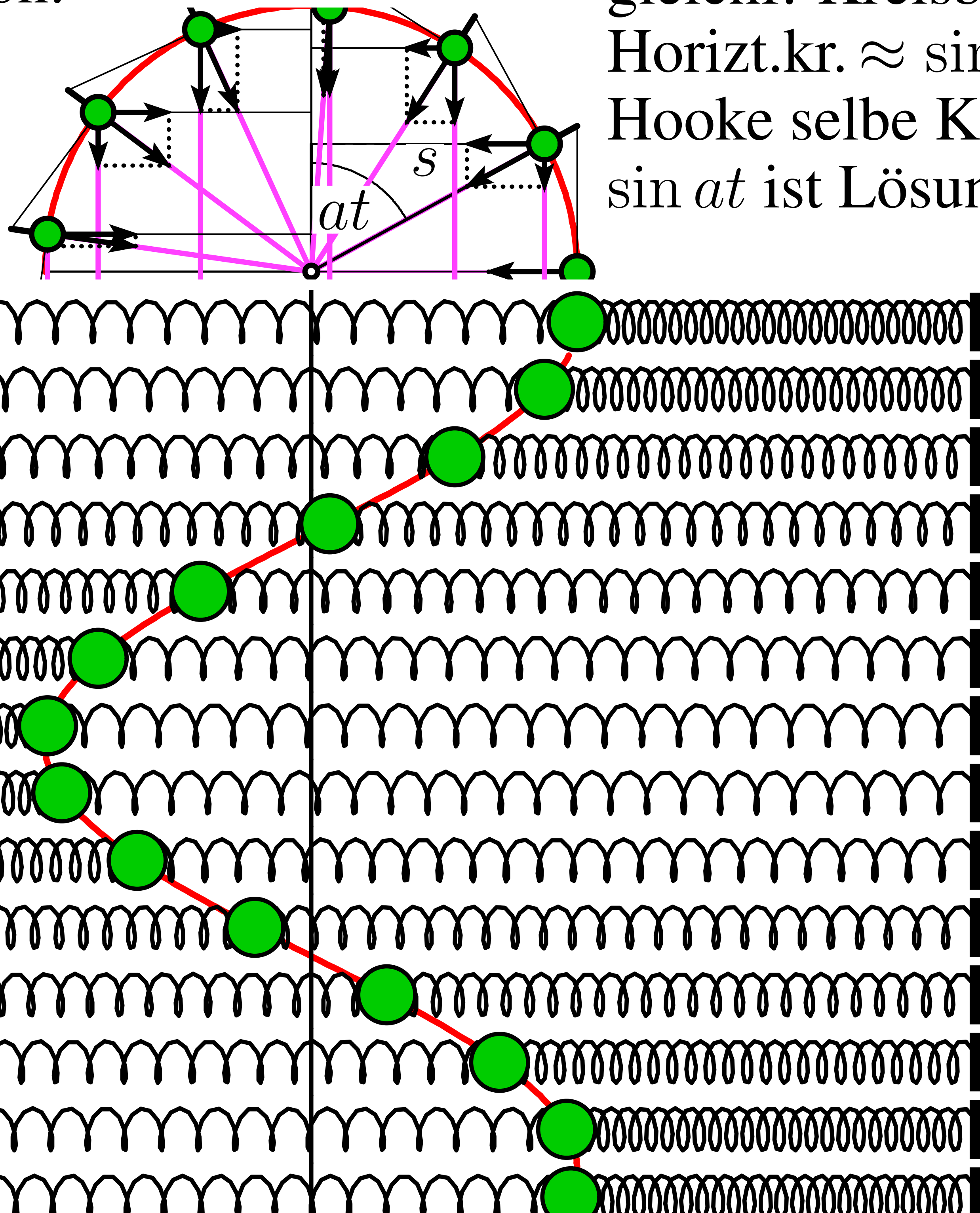
Kraft ist proportional der Auslenkung $f = \text{Const.} \cdot s$

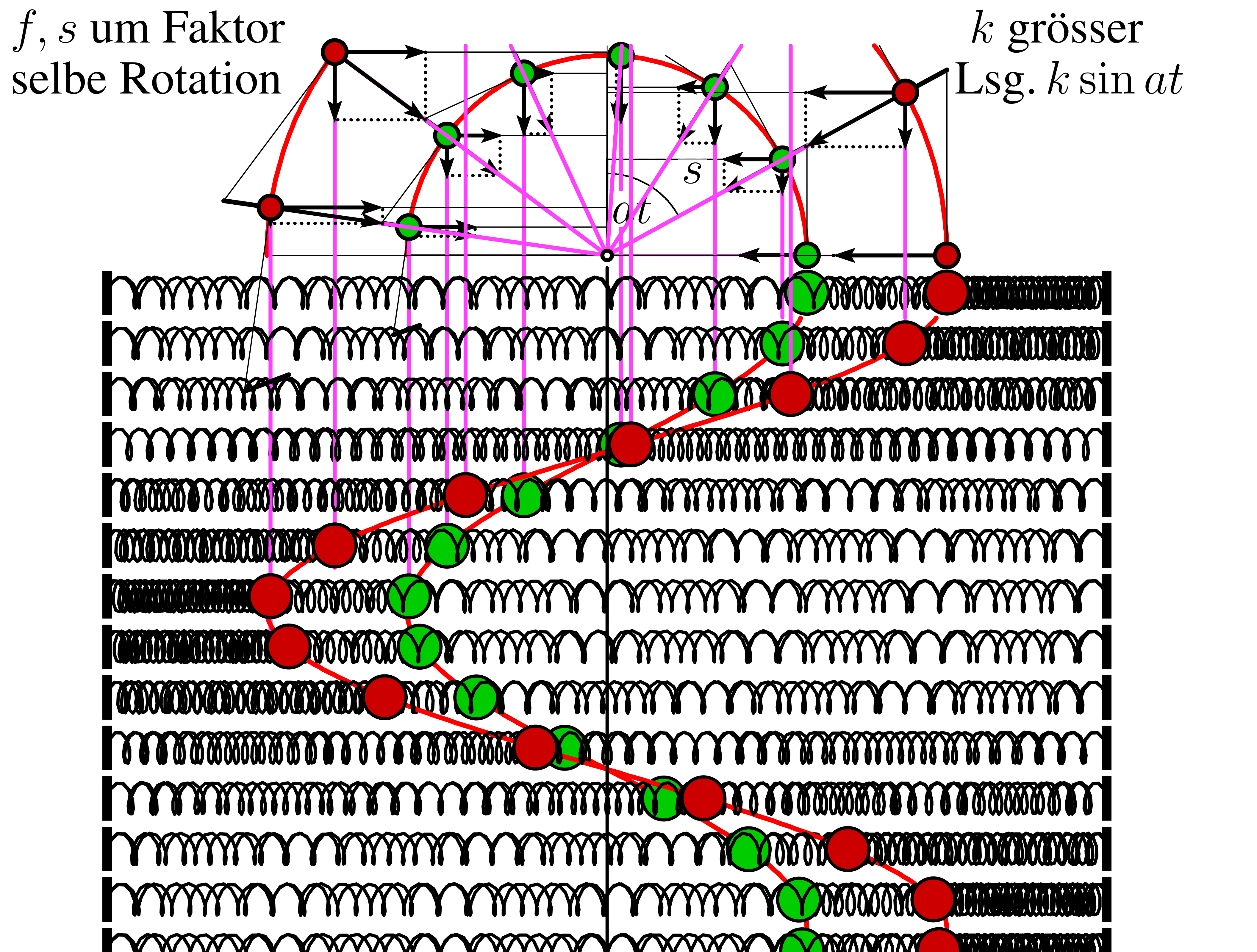
> "The power of any springy body
> is in the same proportion with the extension."

Lsg. Inspir.v.Newton:

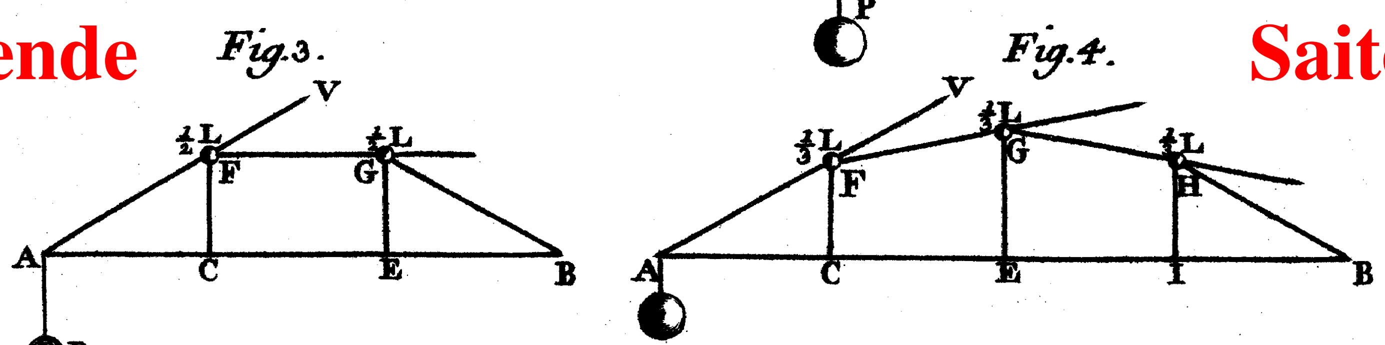


gleichf. Kreisbew.
Horizt.kr. $\approx \sin at$
Hooke selbe Kraft
$\sin at$ ist Lösung

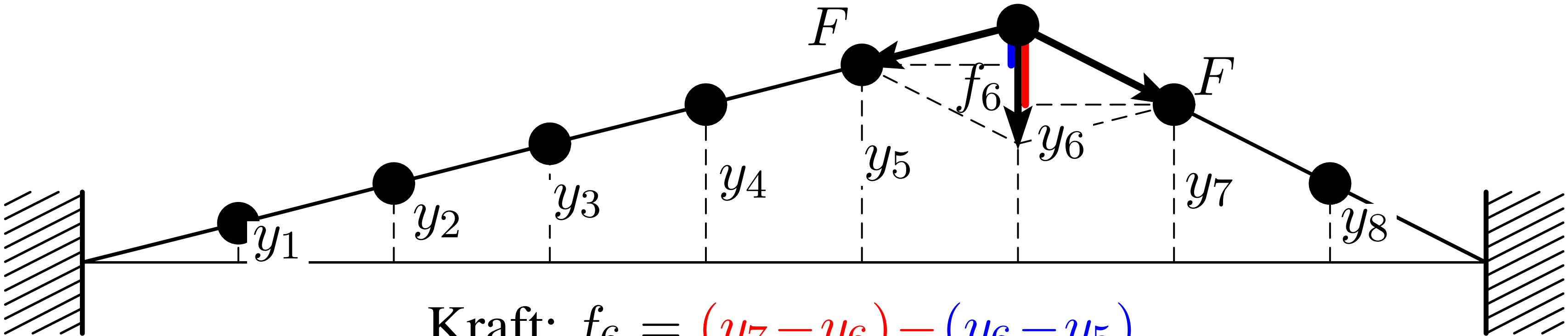



f, s um Faktor
selbe Rotation
k grösser
Lsg. $k \sin at$
s
at

Schwingende Saite



(Abb. von Johann Bernoulli 1728)

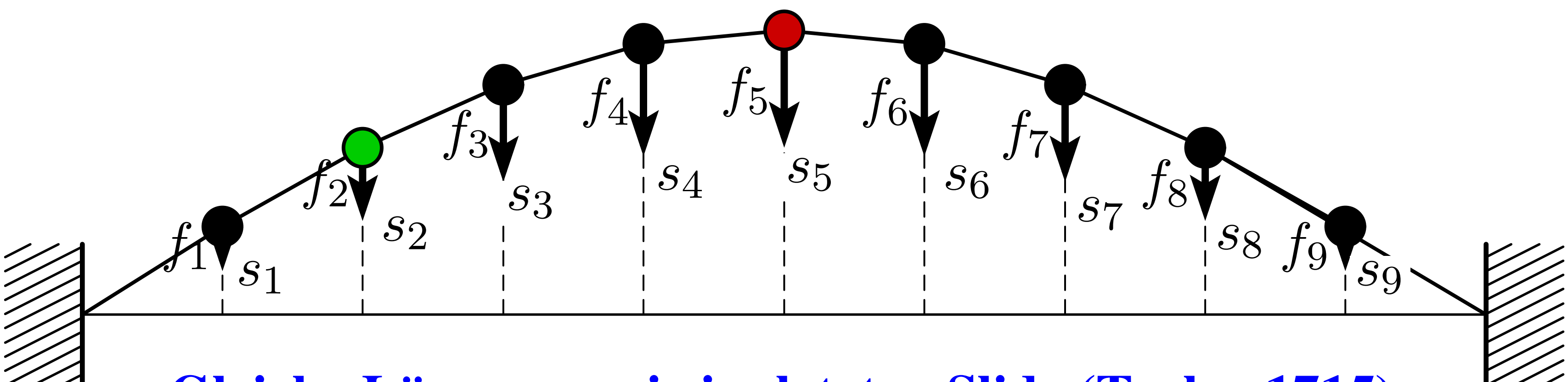


Kraft: $f_6 = (y_7 - y_6) - (y_6 - y_5)$

Bürgi 1: $\sin(x+\delta) - \sin(x-\delta) = 2\cos x \sin\delta$

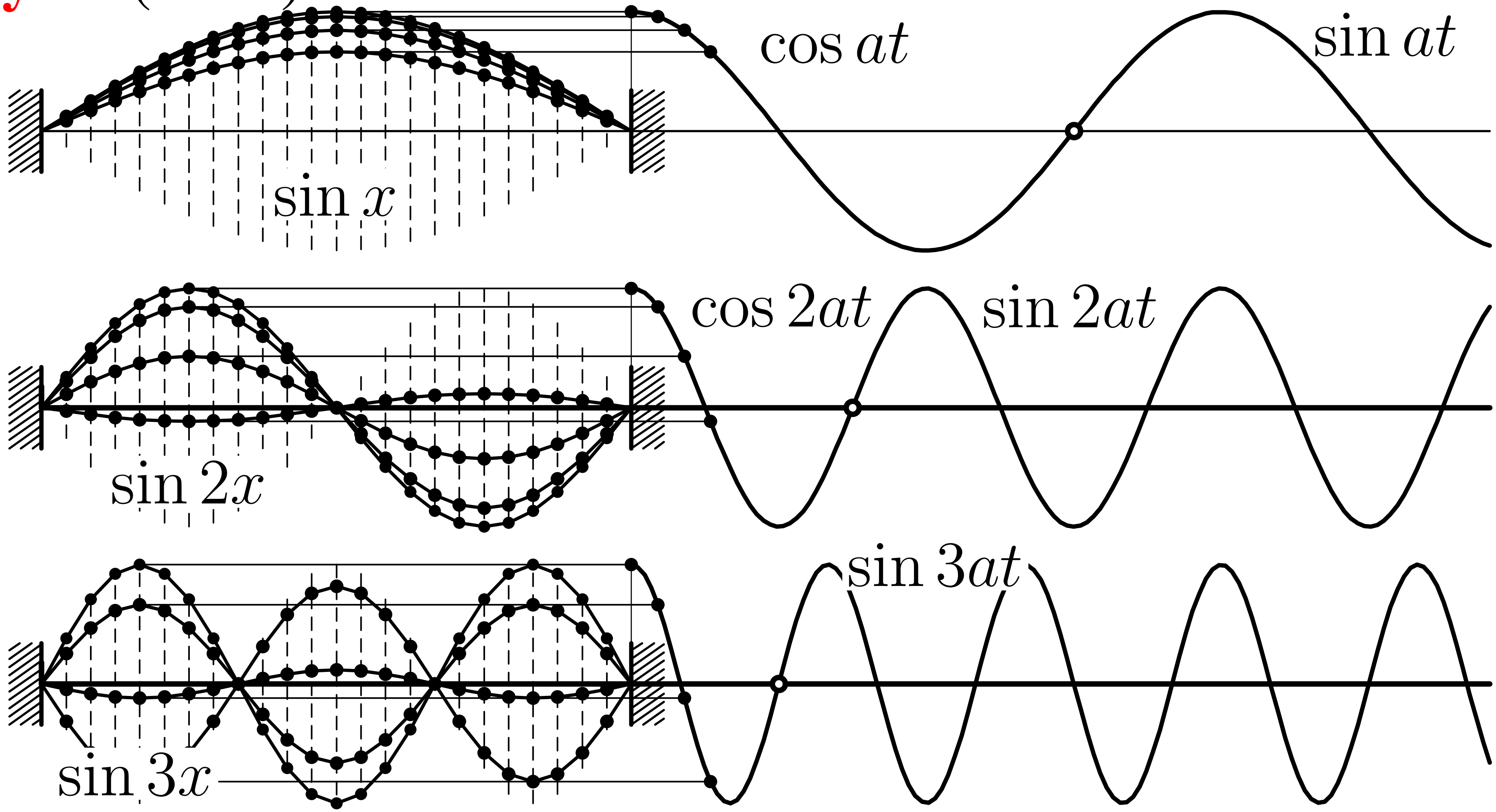
Bürgi 2: $\cos(x+\delta) - \cos(x-\delta) = -2\sin x \sin\delta$

d.h., wenn $y_i = s_i$ Sinuskurve, dann sind Kräfte f_i prop. $-s_i$

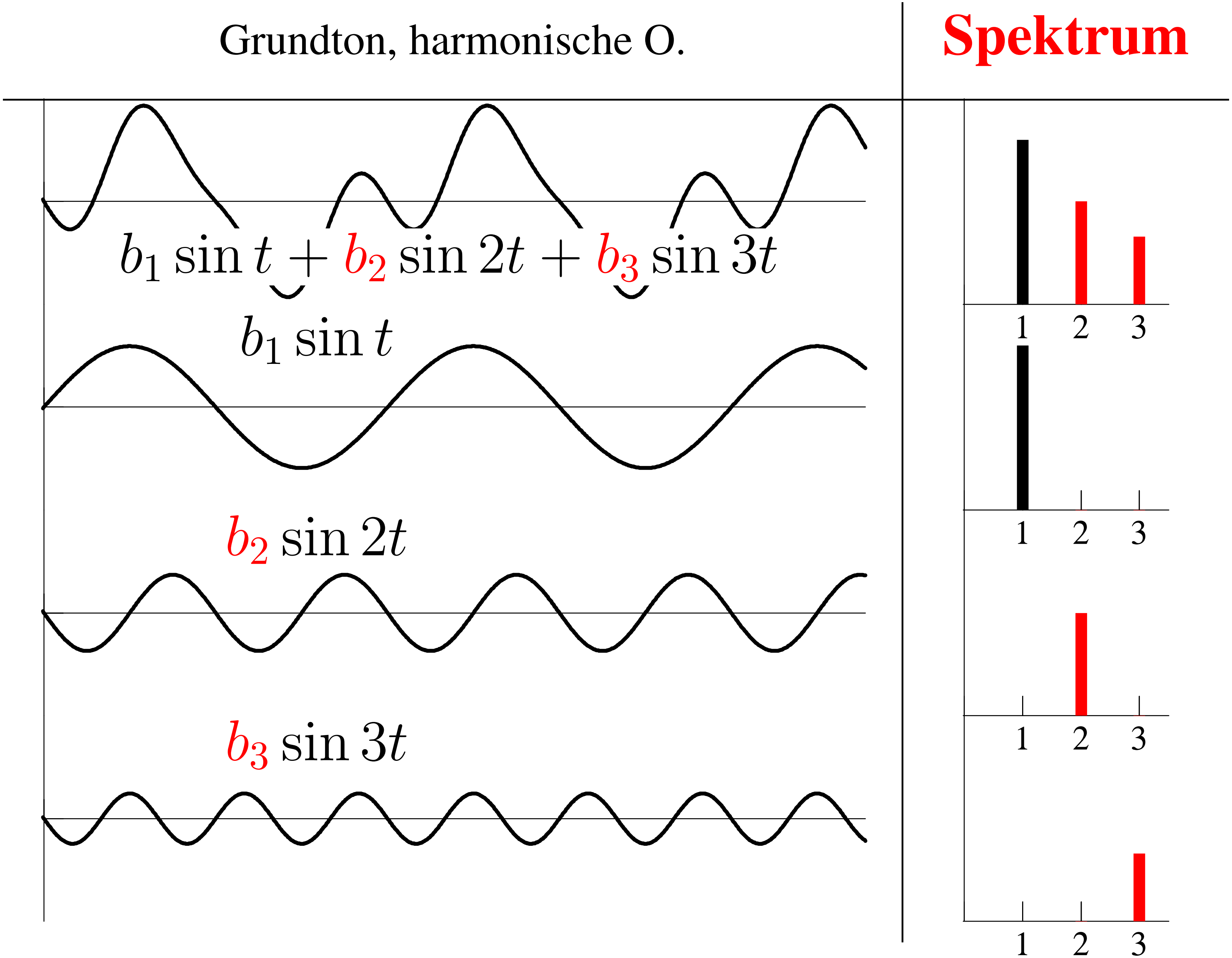


Gleiche Lösungen wie im letzten Slide (Taylor 1715)

Taylor (1715)



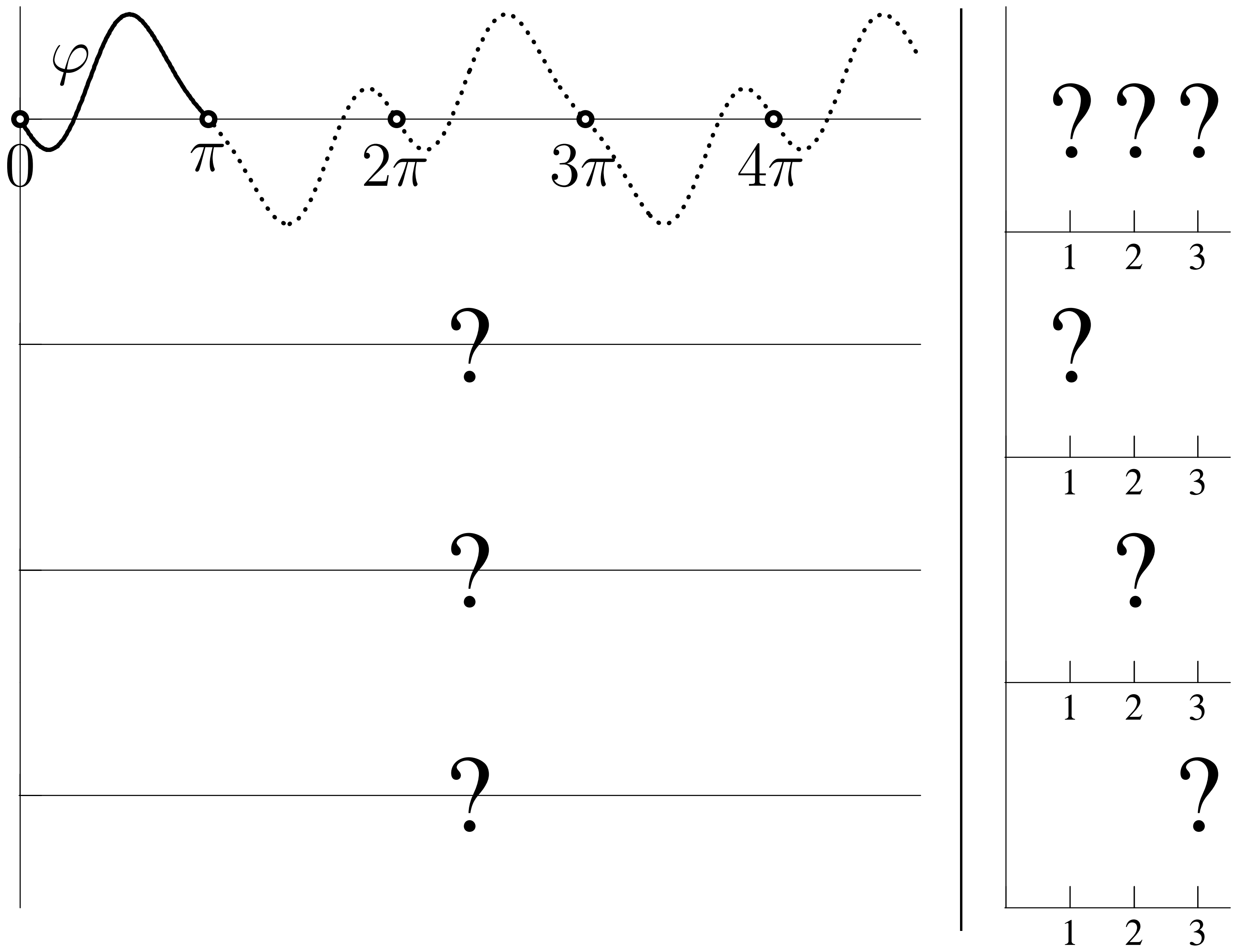
Daniel Bernoulli (1753) Allg. Bew. der Saite: Überlagerung:



Umgekehrtes Problem:

Funktion φ gegeben ($0 \leq t \leq \pi$), suche $b_1, b_2, b_3, \ldots$ sodass

$$\varphi = b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \ldots$$



Lösung(en): 1. In 1764 (**E300**) schrieb Euler kühn

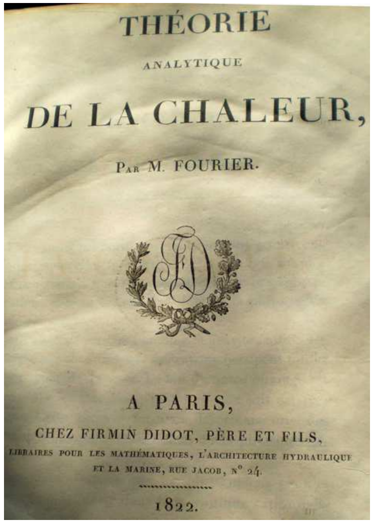
nihil enim impedit,

$$z = A\,\mathrm{sin}.v + B\,\mathrm{sin}.3v + C\,\mathrm{sin}.5v + D\,\mathrm{sin}.7v + E\,\mathrm{sin}.9v + \text{etc.}$$

und schlug zur Berechnung von $A, B, C, ..$ vor, beide Seiten in die Taylorreihe bez. v zu entwickeln und (ohne Bedenken) das erhaltene unendliche lineare Gleichungssystem zu lösen.

Idee wiederentdeckt und ausgeführt von Fourier (1807,1822):

$$1 = a\cos y + b\cos 3y + c\cos 5y + d\cos 7y + \text{\&c. für } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$



THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.

A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

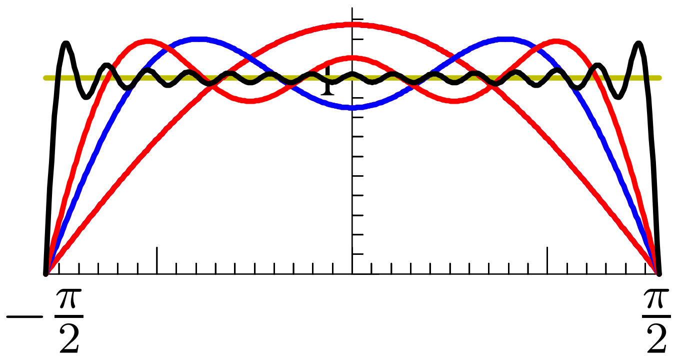
$$\begin{array}{l}
a + b + c + d + e + f + g = 1\\
a + 3^2 b + 5^2 c + 7^2 d + 9^2 e + 11^2 f + 13^2 g = 0\\
a + 3^4 b + 5^4 c + 7^4 d + 9^4 e + 11^4 f + 13^4 g = 0\\
a + 3^6 b + 5^6 c + 7^6 d + 9^6 e + 11^6 f + 13^6 g = 0\\
a + 3^8 b + 5^8 c + 7^8 d + 9^8 e + 11^8 f + 13^8 g = 0\\
a + 3^{10} b + 5^{10} c + 7^{10} d + 9^{10} e + 11^{10} f + 13^{10} g = 0\\
a + 3^{12} b + 5^{12} c + 7^{12} d + 9^{12} e + 11^{12} f + 13^{12} g = 0
\end{array}$$

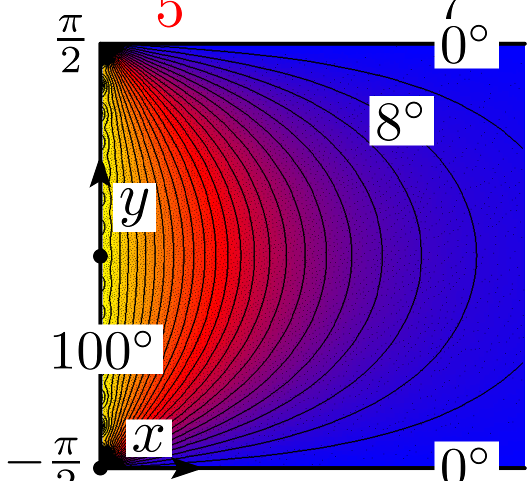
... drei Seiten Rechnung ...

$$a = \frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{9^2}{9^2-1}\cdots = \frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\ldots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\ldots} = \frac{4}{\pi}$$

... vier Seiten Rechnung b, c, d, ...

$$1 = \frac{4}{\pi}\Big[\cos y - \frac{1}{3}\cos 3y + \frac{1}{5}\cos 5y - \frac{1}{7}\cos 7y \pm \text{\&c.}\Big].$$





$\cos y \cdot e^{-x}$

$\cos 3y \cdot e^{-3x}$

$\cos 5y \cdot e^{-5x}$

$\ldots$

Lösung 2. 14 Jahre und 404 Arbeiten nach (**E300**), hatte Euler (**E704**) schliesslich "Nuper autem se mihi obtulit idea ...". Nämlich die Reihe

$$\varphi(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

mit, z.B., $\sin 2x$ multiplizieren und von 0 bis π integrieren

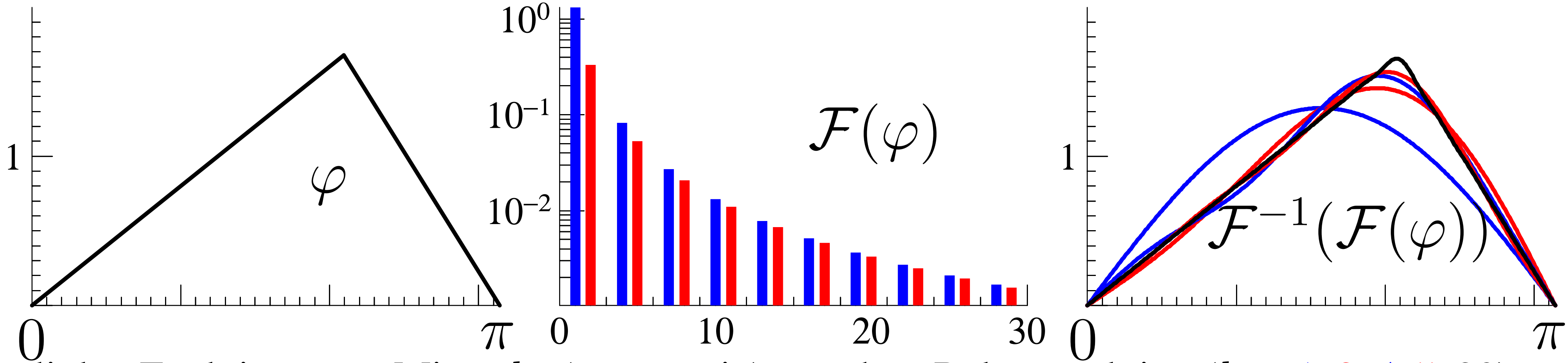
$$\int_0^\pi \varphi(x) \sin 2x \, dx = b_1 \underbrace{\int_0^\pi \sin x \sin 2x \, dx}_{=0} + b_2 \underbrace{\int_0^\pi \sin^2 2x \, dx}_{\pi/2} + b_3 \underbrace{\int_0^\pi \sin 3x \sin 2x \, dx}_{=0} + \ldots$$

(fast alle Integrale sind null, "quod integrale utique evanescit..." siehe Bürgi 2), also $b_2 = \frac{2}{\pi} \int_0^\pi \varphi(x) \sin 2x \, dx$.

Fourier (1807, 1822) hatte unabhängig davon, nach 25 Seiten Rechnung, auch diese Idee:

Fourier Transformation. Für "Signal" $\varphi(x)$, $(0 \leq x \leq \pi)$, gilt für das Spektrum $\mathcal{F}\varphi = \{b_1, b_2, b_3, \ldots\}$

$$\mathcal{F} : \; b_k = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \cdot \sin kx \, dx \qquad \mathcal{F}^{-1} : \; \varphi(x) = \sum_{k=1}^{\infty} b_k \cdot \sin kx \, .$$

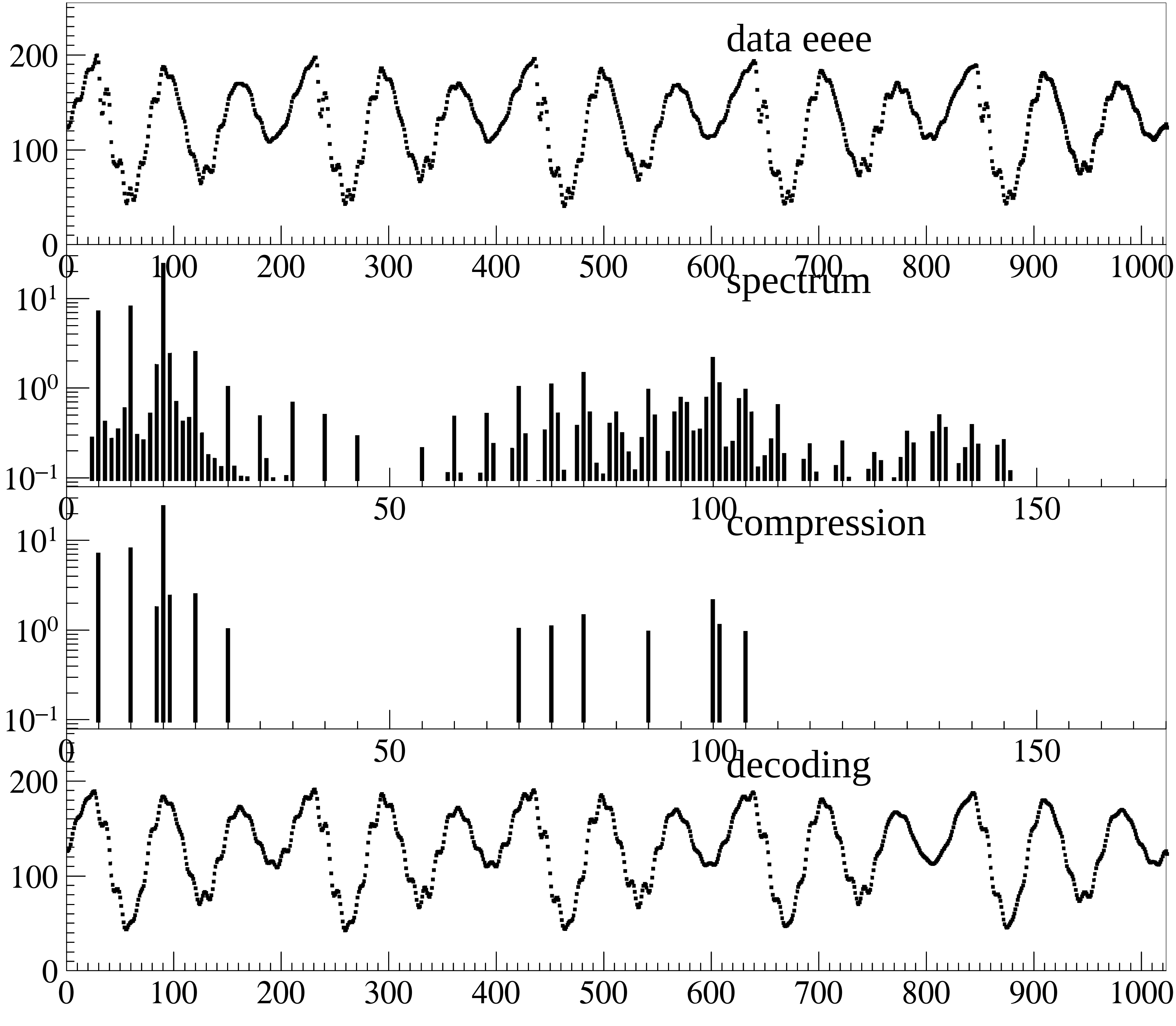


links: Funktion φ; Mitte: b_k (rot negativ); rechts: Rekonstruktion ($k = 1, 2, 4, 5, 29$)

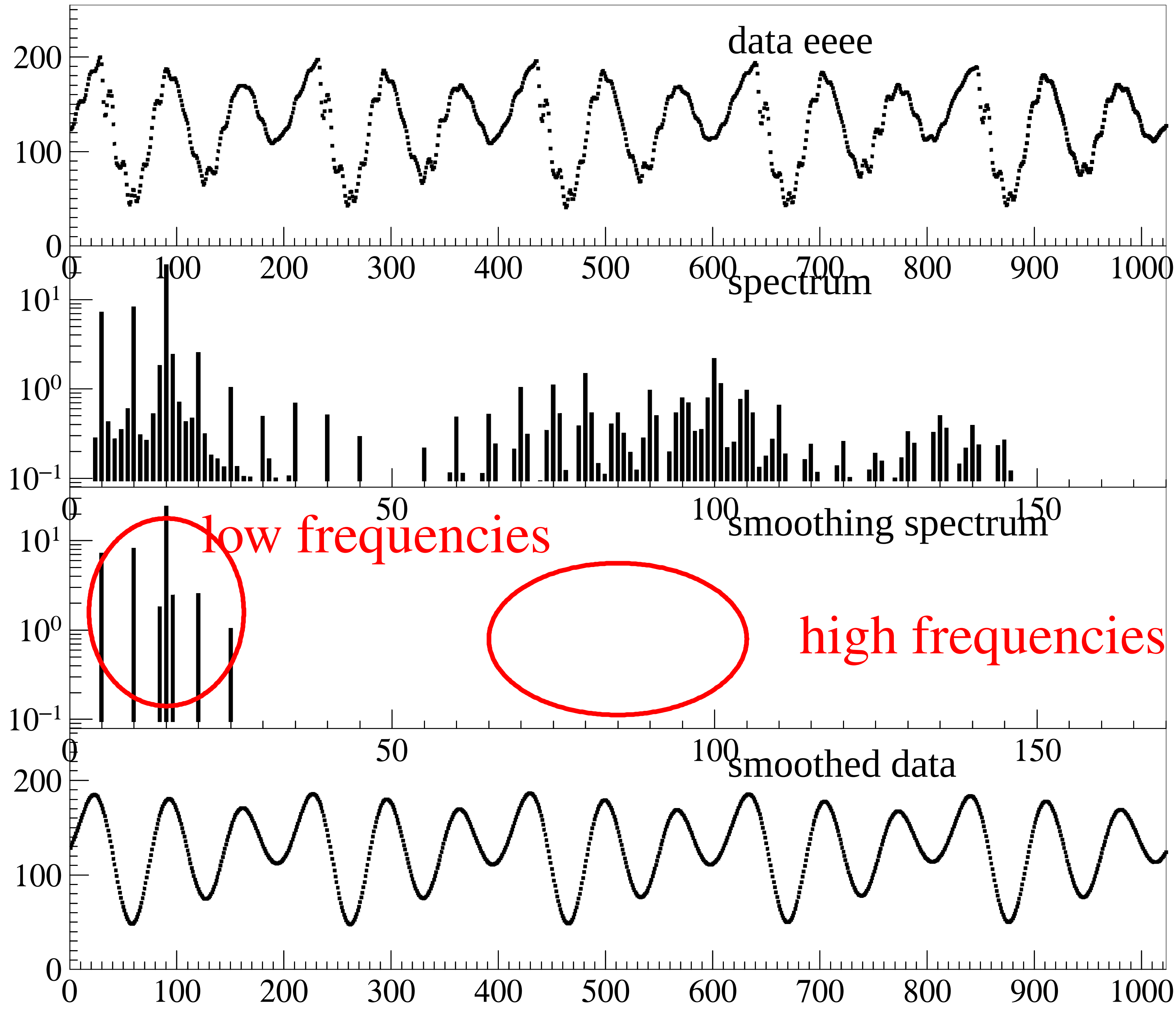
Das Beispiel scheint gut zu funktionieren (selbst bei einem "geknickten" φ), trotzdem gab es für das 19. Jhr. einiges zu klären, z.B.:

- Durfte man die unendliche Reihe gliedweise integrieren (Weierstrass)
- Was ist eigentlich ein "Integral" für die Berechnung von b_k (Riemann)
- Konvergiert die erhaltene Reihe wirklich immer gegen φ (Dirichlet, Fejér)

Principle of data compression (MP3). (Kh. Brandenburg)



Principle of data smoothing, denoising.



Principle of JPEG data compression.

("Joint Photographic Experts Group")

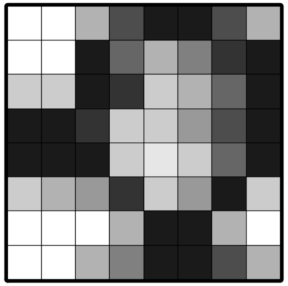
any photograph

is cut into pieces

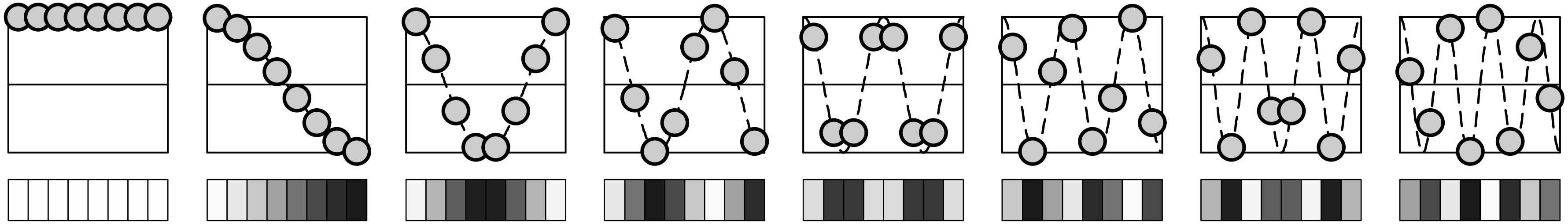
of 8×8 pixel

Principle of JPEG data compression.

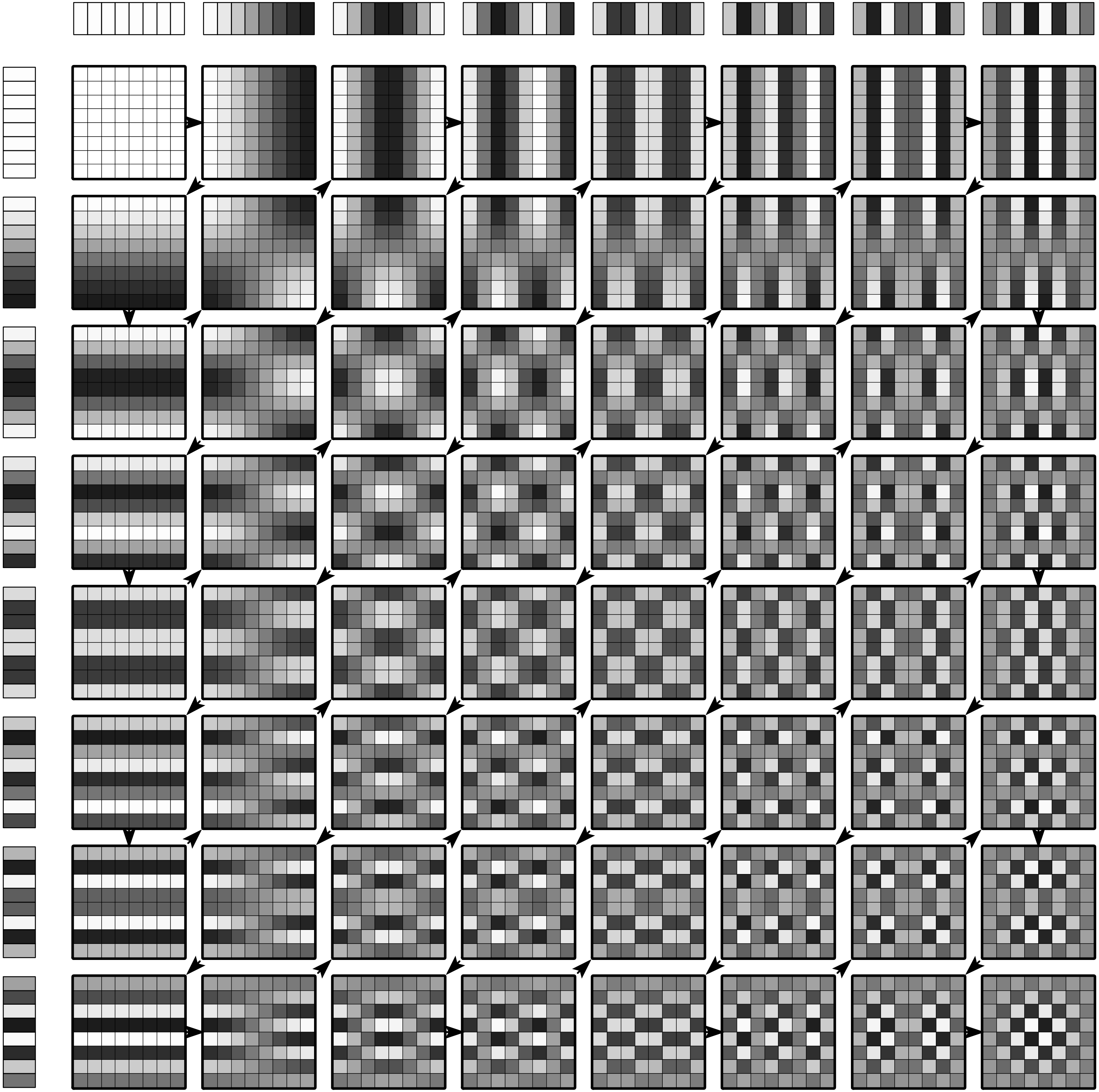
any photograph
is cut into pieces
of 8×8 pixel
$\Rightarrow$
and each piece is
treated individually



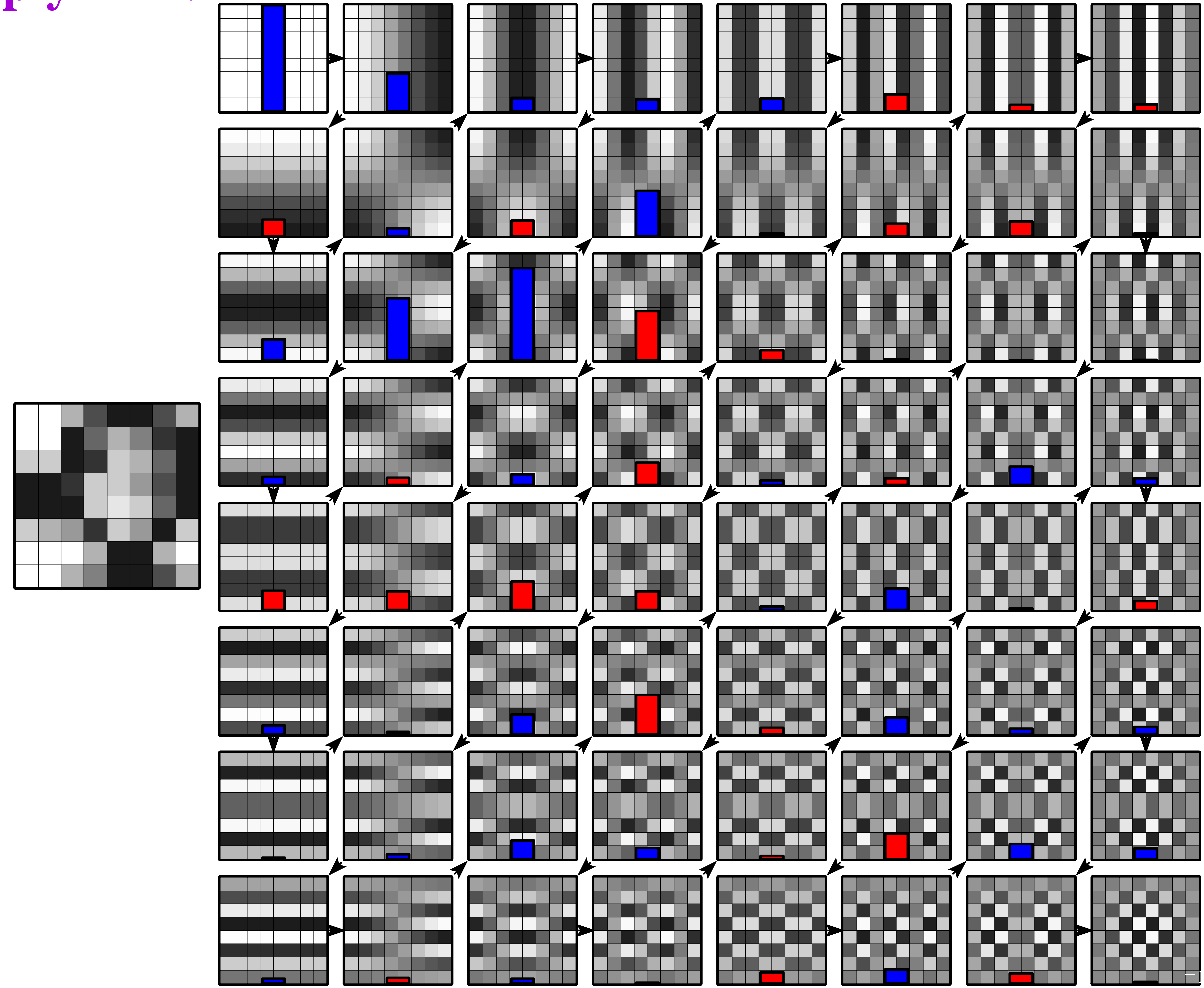
"Sound" is now two-dim.! $\Rightarrow$ one-dim. base $\cos k\pi x$



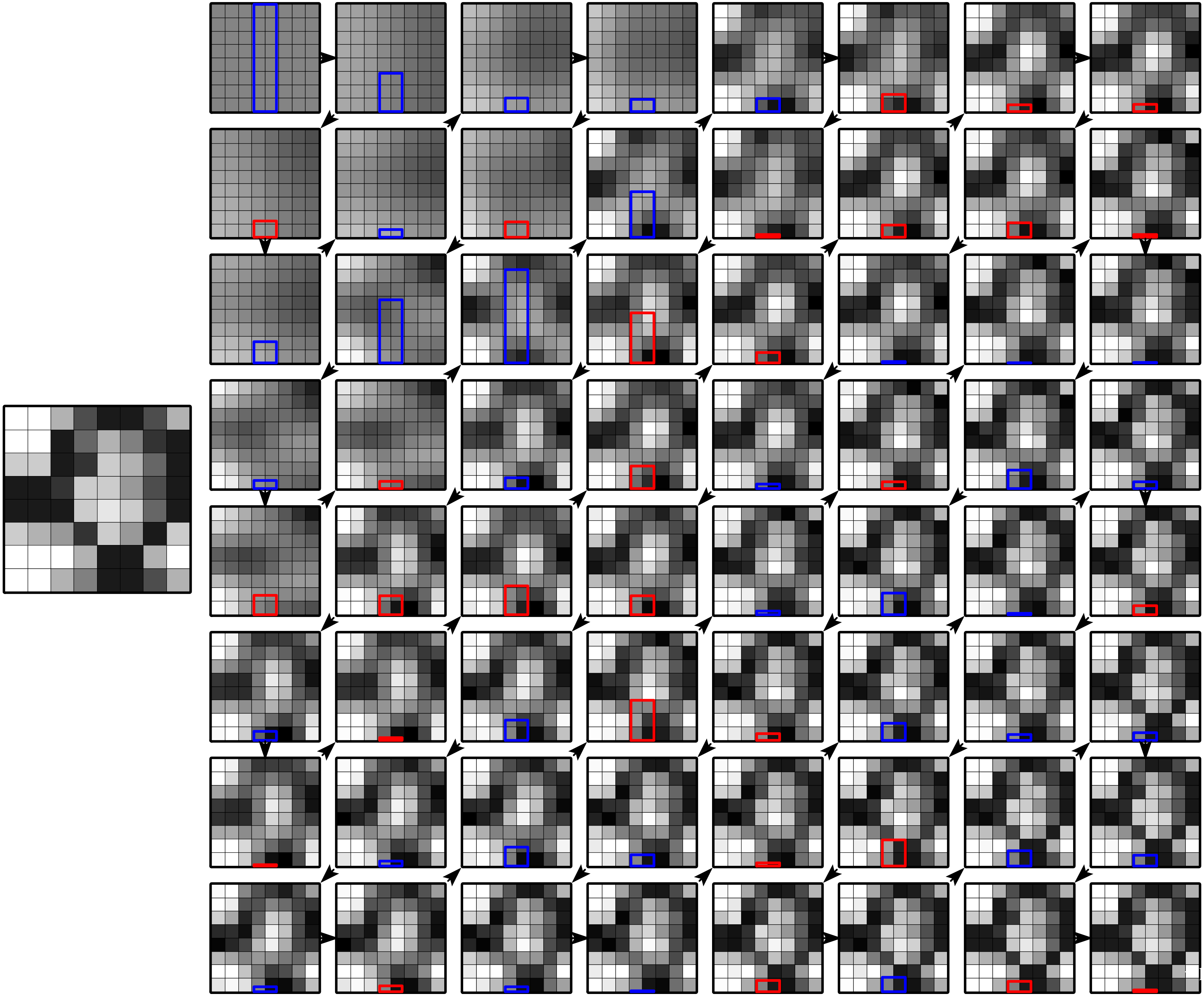
transform into two-dimensional base... $\cos k\pi x \cdot \cos \ell\pi y \Rightarrow$



Apply FFT:



stages of reconstruction of image :



• Adaptation à des films et DVDs:

JPEG ⇒ MovingPictureExpertsGroup ⇒ MP3

• CDs encode sound as linear PulseCodeModulation, hence, not compressed [1981 technology] !

• Transferring CD tracks to something like an Apple iPod (portable music device) will compress the results via FFT ;

• Audio tracks on DVDs use a similar process, compressing sound frequency data (Kyle Granger).

Vielen Dank.

Trugarez   Grazie tanto   Thank you   Arigatō   Merci   Kiitos

Mange Tak   Tack så mycket   Muchas gracias   Спасибо

谢 谢

Auf Wiedersehen am 22.02.2022 zu 200 Jahre *Théorie de la Chaleur* (Fourier).