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6. Exercise sheet for Numerik für Differentialgleichungen auf Oberflächen

Exercise 12. Let the evolving surface $\Gamma(t)$ be given by the following distance function

$$d(x,t) = \frac{x_1^2}{a(t)} + x_2^2 + x_3^2 - 1^2,$$

where $a(t) = 1 + \sin(2\pi t)/4$.

(a) Compute the normal vector $\nu_{\Gamma(t)}$ and the normal velocity v.

(b) What is the evolution of the surface over the time interval $[0,1]$?

(c) How would you visualise this evolving surface? Write a simple pseudo-code.

Exercise 13. Prove the first Leibniz formula from below. Let u be a function (such that all the following quantities exist), then the following equalities hold

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u = \int_{\Gamma(t)} \partial^\bullet u + \int_{\Gamma(t)} u(\nabla_{\Gamma(t)}\cdot v)$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)}|\nabla_{\Gamma(t)}u|^2 = \int_{\Gamma(t)}\nabla_{\Gamma(t)}u\cdot\nabla_{\Gamma(t)}\partial^\bullet u + \frac{1}{2}\int_{\Gamma(t)}|\nabla_{\Gamma(t)}u|^2(\nabla_{\Gamma(t)}\cdot v) - \int_{\Gamma(t)}\mathcal{D}(v)\nabla_{\Gamma(t)}u\cdot\nabla_{\Gamma(t)}u,$$

where $\mathcal{D}(v) = \frac{1}{2}\big(\nabla_{\Gamma(t)}v + (\nabla_{\Gamma(t)}v)^T\big)$.

Hint. Recall the following definitions and facts. Let $X(\cdot,t) : \Omega \to \Gamma(t)$ be a parametrisation from $\Omega \subset \mathbb{R}^2$ (note that $\Omega \neq \Gamma^0$), of the surface $\Gamma(t)$ evolving according to the ODE $\partial_t X(\vartheta,t) = v(X(\vartheta,t),t)$. The Riemannian metric and related quantities are

$$g_{ij} = \partial_{\vartheta_i}X\cdot\partial_{\vartheta_j}X, \quad g = \det(g_{ij}), \quad g^{ij} = (g_{ij})^{-1}.$$

For a function $u(\cdot,t) : \Gamma(t) \to \mathbb{R}$ we define $U(\vartheta,t) = u(X(\vartheta,t),t)$. Tangential gradient and divergence are given by

$$(\nabla_\Gamma u)(X(\vartheta)) = \sum_{i,j} g^{ij}\partial_{\vartheta_i}X\partial_{\vartheta_j}U \quad \text{and} \quad (\nabla_\Gamma\cdot v)(X(\vartheta)) = \sum_{i,j} g^{ij}\partial_{\vartheta_i}X\cdot\partial_{\vartheta_j}V.$$

An integral over a Riemmanian manifold is defined by

$$\int_{\Gamma(t)} u = \int_\Omega U(\vartheta,t)\sqrt{g}\,\mathrm{d}\vartheta.$$

The determinant satisfies the Euler relation

$$\partial_t\sqrt{g} = \sqrt{g}\sum_{i,j} g^{ij}\partial_{\vartheta_i}X\cdot\partial_{\vartheta_j}V.$$

Exercise 14. For a weak solution u, with initial value $u^0 \in H^1(\Gamma^0)$ and $f = 0$, the energy equations hold:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} |u|^2 + \int_{\Gamma(t)} |\nabla_{\Gamma(t)} u|^2 = -\frac{1}{2}\int_{\Gamma(t)} |u|^2 (\nabla_{\Gamma(t)} \cdot v)$$

$$\int_{\Gamma(t)} |\partial^\bullet u|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} |\nabla_{\Gamma(t)} u|^2 = \frac{1}{2}\int_{\Gamma(t)} |\nabla_{\Gamma(t)} u|^2 (\nabla_{\Gamma(t)} \cdot v) - \int_{\Gamma(t)} \mathcal{D}(v) \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} u - \int_{\Gamma(t)} u \partial^\bullet u (\nabla_{\Gamma(t)} \cdot v),$$

with $\mathcal{D}(v) = \frac{1}{2}\big(\nabla_{\Gamma(t)} v + (\nabla_{\Gamma(t)} v)^T\big)$.

Prove the first energy equation using the weak formulation and the Leibniz formula. (Try the second one as an extra exercise, if you like.)

Exercise 15. Prove that for a weak solution u, with initial value $u^0 \in H^1(\Gamma^0)$ and $f = 0$, the a priori estimates also hold, for $t_0 \in [0, T]$,

$$(a) \qquad \sup_{t \in (0,t_0)} \|u(\cdot,t)\|^2_{L^2(\Gamma(t))} + \int_0^{t_0} \|\nabla_{\Gamma(t)} u(\cdot,t)\|^2_{L^2(\Gamma(t))} \mathrm{d}t \leq c\|u^0\|^2_{L^2(\Gamma^0)},$$

$$(b) \qquad \int_0^{t_0} \|\partial^\bullet u(\cdot,t)\|^2_{L^2(\Gamma(t))} \mathrm{d}t + \sup_{t \in (0,t_0)} \|\nabla_{\Gamma(t)} u(\cdot,t)\|^2_{L^2(\Gamma(t))} \leq c\|u^0\|^2_{H^1(\Gamma^0)}.$$

Hint. Use the energy equations, then estimate the right-hand sides, and use Gronwall's inequality.

Programming exercise 2. Consider the parabolic problem on Γ the sphere of unit radius:

$$\begin{aligned} \partial_t u - \Delta_\Gamma u &= f && \text{on} \quad \Gamma \quad \text{for} \quad [0,1], \\ u(\cdot,0) &= u^0 && \text{on} \quad \Gamma. \end{aligned}$$

(a) Assume that the exact solution is given to be, with $x = (x_1, x_2, x_3)$,

$$u(x,t) = e^{-t} x_1 x_2.$$

Compute the right-hand side function corresponding to the PDE. Create the functions `func_sol.m` and `func_f.m` with them. (Use vector operations!)

(b) Approximate the above problem using surface finite elements as a space discretisations, combined with efficient time integrators (given below).

To assemble the mass and stiffness matrices use the already implemented function `[A,M]=surface_assembly(Elements,Nodes)` from PA1.

Implement the time discretisations:

- general Runge–Kutta methods (given by its Butcher tableau in `Butcher_tableau.m`);
- and BDF methods of order $k \leq 5$ (given by its coefficients in `BDF_tableau.m`).

Use the exercises 10 and 11. As initial data use the (SFEM) interpolation of the exact solution.

(c) Use the 3 stage Radau IIA method and the 3-step BDF method to solve the corresponding fully discretised problem over the time interval $[0,1]$. Using all meshes from PA1

`Sphere_elements_j.txt`
`Sphere_nodes_j.txt` $\quad j = 0, \ldots, 5,$

and all the time step sizes $\tau = 0.2, 0.1, 0.05, 0.025, 0.0125$.

Compute the following errors of the numerical solution, when compared to the exact solution given by `func_sol.m` from above.

As an output generate two convergence plots* using the errors measured in the L^2 norm and the H^1 semi-norm at step N (with $N\tau = 1$):

$$\begin{aligned} \|(u_h^N)^\ell - u(t_N)\|_{L^2(\Gamma)}^2 &\approx \|\boldsymbol{e}^N\|_{\boldsymbol{M}}^2 = (\boldsymbol{e}^N)^T \boldsymbol{M} \boldsymbol{e}^N, \\ \|\nabla((u_h^N)^\ell - u(t_N))\|_{L^2(\Gamma)}^2 &\approx \|\boldsymbol{e}^N\|_{\boldsymbol{A}}^2 = (\boldsymbol{e}^N)^T \boldsymbol{A} \boldsymbol{e}^N. \end{aligned}$$

- In the first figure plot the error curves for each mesh against the step size.
- In the second figure plot the error curves for each time step size against the mesh width.

* As discussed before. Also see the example files (for second order finite elements)!

Bonus question: Explain the behaviour of the convergence plots. Does the "flatting out" of the curves contradict the convergence theory?

The functions and the grid arrays can be found at `https://na.uni-tuebingen.de/ex/surfPDE_ss18/PA2.zip`.

Discussed on the tutorials on 12.06.2018. The programming exercise is due on 19.06.2018, 12 s.t.