Universität Tübingen
Mathematisches Institut
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Tübingen, July 8, 2026

12. Sheet for Numerics of Instationary Differential Equations

Exercise 31:

Let $V(x)$ be a real potential. Prove that the system

$$i\partial_t u(t,x) = V(x)u(t,x)$$

preserves the modulus in time, i.e. $|u(t,x)| = |u(0,x)|$.
Furthermore, show that

$$u(t,x) = e^{-itV(x)}u(0,x).$$

Exercise 32:

Consider the free Schrödinger equation

$$i\partial_t u(t,x) = -\Delta u(t,x)$$

1) Show that this equation generates a strongly continuous unitary (semi-)group $U(t) = e^{it\Delta}$ on $L^2(\mathbb{R}^n)$. I.e. show:
   (a) Group property: $U(t)U(s) = U(t+s)$.
   (b) Identity: $U(0) = I$.
   (c) Unitary property: $U(t)^*U(t) = I$.
   (d) Strong continuity: For any $u \in L^2(\mathbb{R}^n)$, $\lim_{t\to 0} \|U(t)u - u\|_{L^2} = 0$.
2) Prove that $\|U(t)\| = 1$ for all $t \in \mathbb{R}$.

Hint: You may use the Fourier transform to diagonalize the operator.

Exercise 33:

Prove that, if $u \in C^{m+2}$ and $Vu \in C^m$ with $m \geq 2$, then for $t \geq 0$:

$$\|u_N(\cdot,t) - u(\cdot,t)\|_{L^2(-\pi,\pi)} \leq CN^{-m}(1+t).$$

where, u_N is the solution of the Fourier collocation for the Schrördinger equation. To do so:

(a) define $\widetilde{u} := I_N(u)$ and split the error.

(b) find out what $i\partial_t\widetilde{u}$ is equal to using the fact that I_N is independent of time, and then the corresponding equation for the error part $\|u_N - \widetilde{u}\|$ and its defect.

(c) To get the final error bounds use energy estimates and the aliasing formula.

Solutions are discussed on Wednesday July 15, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.