Universität Tübingen
Mathematisches Institut
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Tübingen, June 30, 2026

11. Sheet for Numerics of Instationary Differential Equations

Exercise 28: Is the initial value problem ($x \in \mathbb{R}$, $t \geq 0$)

$$u_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_x + Bu, \quad u|_{t=0} = u_0$$

with a constant matrix $B \in \mathbb{R}^{2\times 2}$ well-posed?

Exercise 29:

Show that the Lax Wendroff method

$$\frac{u_j^{n+1} - u_j^n}{\tau} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{c^2\tau}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$$

with the numerical boundary condition

$$\frac{u_0^{n+1} - u_0^{n-1}}{2\tau} = c\frac{u_1^n - u_0^n}{h}$$

is instable.

Exercise 30:

Show that the Leapfrog-method

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

together with the boundary condition

$$\frac{u_0^{n+1} - u_0^n}{\tau} = c\frac{u_1^n - u_0^n}{h}$$

is stable. For this, consider the associated symbols

$$a(z,\xi) = z - z^{-1} - r(\xi - \xi^{-1}),$$
$$b(z,\xi) = z - 1 - r(\xi - 1)$$

(where $r = c\tau/h$ is the Courant number) and proceed the following:

(a) For $|z| > 1$, there exists exactly one zero $\xi_1(z)$ of $a(z,\xi) = 0$ with absolute value smaller than 1 and this zero satisfies $\lim_{z\to 1} \xi_1(z) = -1$.

Hint: Show that for real $z \in (1,\infty)$, there is one zero $\xi_1(z) \in (0,1)$ and one zero $\xi_2(z) \in (1,\infty)$. Then, consider $z = e^{\alpha}e^{i\varphi}$.

(b) The expansion

$$\frac{1}{b(z,\xi_1(z))} = \sum_{n=0}^{\infty} b_n z^{-n}, \quad |z| > 1$$

has a bounded coefficient sequence (b_n).

Hint: Show, that $b(z,\xi_1(z)) \neq 0$ for $|z| > 1$. Then, compute the Laurent series of $\frac{1}{b(z,\xi_1(z))}$, whose non-principal part vanishes. The method is therefore stable.

Solutions are discussed on Tuesday July 7, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.