

10. Sheet for Numerics of Instationary Differential Equations

Exercise 26:

Consider the Crank–Nicolson method applied to the heat equation $u_t = u_{xx}$

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{2} \left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \right).$$

- (a) Derive the method.
- (b) Determine the growth factor

$$G(\alpha) = \frac{P(e^{ih\alpha})}{Q(e^{ih\alpha})}$$

- (c) Study the stability, that is, determine a γ independent of h, τ and α such that

$$|G(\alpha)| \leq e^{\gamma\tau}.$$

- (d) Find a p such that

$$|G(\alpha) - e^{-\tau\alpha^2}| \leq C\tau h^p (1 + |\alpha|^q),$$

where C and q are independent of h, τ and α .

Exercise 27:

As in the previous exercise, formulate the Crank–Nicolson method for the Schrödinger equation $u_t = iu_{xx}$. Study the growth factor $G(\alpha)$, stability and order of the method. In addition, show the conservation of the ℓ^2 -norm:

$$\sum_{j=-\infty}^{\infty} |u_j^{n+1}|^2 = \sum_{j=-\infty}^{\infty} |u_j^n|^2.$$

Solutions are discussed on Tuesday July 1, 2026

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