Universität Tübingen
Mathematisches Institut
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9. Sheet for Numerics of Instationary Differential Equations

Exercise 23: Let V be a separable Hilbert space with norm $\|\cdot\|$ and corresponding inner product $(\cdot,\cdot)$. Prove: For a sequence of Fourier coefficients $\{u_n\}_n \subset V$ defined by

$$u_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi}\widehat{u}(\varphi)\,d\varphi, \qquad \widehat{u}(\varphi) = \sum_{n=0}^{\infty} u_n e^{in\varphi}.$$

Parseval's theorem holds:

$$\sum_{n=0}^{\infty} \|u_n\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|\widehat{u}(\varphi)\|^2\,d\varphi.$$

Exercise 24: Solve the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ on $[0,\pi]$ with initial values

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x)$$

and homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0$$

using Fourier series. For this, assume that the exact solution u exists and extend it symmetrically to $[-\pi,0]$. Prove: if u_0 and v_0 are real-valued, then u is real-valued.

Exercise 25: Consider the differential equation $u_t = cu_x$ and the Lax-Friedrichs method

$$\frac{u_j^{n+1} - \frac{1}{2}[u_{j+1}^n + u_{j-1}^n]}{\tau} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

as well as the Lax-Wendroff method

$$\frac{u_j^{n+1} - u_j^n}{\tau} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{c^2\tau}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

For the initial value $u(x,0) = \exp(i\alpha x)$, determine the growth factor $G(\alpha)$ and do a von Neumann stability analysis, that is, formulate a condition for $c\tau/h$, such that $|G(\alpha)| \le 1$ for all $\alpha \in \mathbb{R}$.

Programming Exercise 2:

Implement the two methods of the previous exercise for the one-dimensional problem

$$\begin{aligned} u_t(x,t) &= cu_x(x,t), && x \in [x_{\min}, x_{\max}],\ t > 0,\\ u(x,0) &= \alpha\exp(-\beta(x-\gamma)^2), && x \in [x_{\min}, x_{\max}]. \end{aligned}$$

- Use as boundary conditions

$$u(x_{\min},t) = \alpha\exp(-\beta(x_{\min}+ct-\gamma)^2), \qquad u(x_{\max},t) = \alpha\exp(-\beta(x_{\max}+ct-\gamma)^2).$$

- Experiment with different values of α, β and γ and make clear how these parameters affect the solutions. Write a short comment.
- Test different values of c. In which direction the wave is transported?

- Experiment with different values of c, τ and h to see whether the numerical solution is bounded or not. How does it coincide with the von Neumann stability analysis of exercise 25?
- Create convergence plots for h, for instance using $c = -0.5$, $\tau = \frac{1}{160}$ and

$$h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}.$$

  Integrate until $N\tau = t_{\text{end}} = 0.5$ and plot the errors $\max_j |u(x_j, t_{\text{end}}) - u_j^N|$. As parameters, choose

$$\alpha = 1, \quad \beta = 10, \quad \gamma = 0.2, \quad x_{\min} = 0, \quad x_{\max} = 1.$$

Solutions are discussed on Tuesday June 24, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.