Universität Tübingen
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8. Sheet for Numerics of Instationary Differential Equations

Exercise 21:

(a) Show per induction for j, that the sequence $y_k = \zeta^k$, $k = 0, 1, \dots$ satisfies:

$$\nabla^j y_k = \zeta^k \left(1 - \frac{1}{\zeta}\right)^j,$$

where $\nabla^0 y_k = y_k$, $\nabla^j y_k = \nabla^{j-1} y_k - \nabla^{j-1} y_{k-1}$ for $j \geq 1$.

(b) Using this, show that for BDF methods (given by $\sum_{j=1}^k j^{-1} \nabla^j y_{n+k} = h f_{n+k}$):

$$\alpha(\zeta) = \zeta^k \sum_{j=1}^k \frac{1}{j} \left(1 - \frac{1}{\zeta}\right)^j, \qquad \beta(\zeta) = \zeta^k.$$

(c) Formulate the BDF semi-discretization in time for the parabolic problem $\partial_t u + \Delta u = f(t)$.

Exercise 22: (Adaptive step sizes for Runge–Kutta-Methods)

For adaptive step sizes one uses embedded methods of the form

$$\widehat{y}_1 = y_0 + h \left(\widehat{b}_0 f(t_0, y_0) + \sum_{j=1}^s \widehat{b}_j Y_j' \right) = y_0 + \left(h\widehat{b}_0 f(t_0, y_0) + \sum_{j=1}^s \widehat{d}_j Z_j \right)$$

with the same nodes c_i but of lower order (for Radau: order s). Hence, we have

$$\widehat{y}_1 - y_1 = h\widehat{b}_0 f(t_0, y_0) + \sum_{j=1}^s h(\widehat{b}_j - b_j) Y_j' = \left(h\widehat{b}_0 f(t_0, y_0) + \sum_{j=1}^s (\widehat{d}_j - d_j) Z_j \right),$$

where $Z_j = Y_j - y_0$ and $d = b^\top A^{-1}$.

(a) Make clear: With adequate choice of $\widehat{d}_j$ the error $\mathrm{err} := \widehat{y}_1 - y_1$ fulfills

$$\|\mathrm{err}\| = C\, h^{s+1} + O(h^{s+2}).$$

(b) Applying this error bound to the test equation $y' = \lambda y$, $y(0) = y_0$, for $h\lambda \to \infty$, the error bound behaves like $\widehat{b}_0 h \lambda y_0$ (why?) and therefore is not useful for stiff differential equations. If one uses

$$\mathrm{err} := (I - h\widehat{b}_0 J)^{-1} (\widehat{y}_1 - y_1), \tag{1}$$

then $\mathrm{err} \to -y_0$ for $h\lambda \to \infty$, where $J = \lambda I$ is the Jacobi matrix of the test equation. In the first and every rejected step ($\|\mathrm{err}\| > 1$) we set

$$\widehat{\mathrm{err}} := (I - h\widehat{b}_0 J)^{-1} \left(h\widehat{b}_0 f(t_0, y_0 + \mathrm{err}) + \sum_{j=1}^s (\widehat{d}_j - d_j) Z_j \right).$$

With this we get $\widehat{\mathrm{err}} \to 0$ for $h\lambda \to \infty$ as for the numerical solution. Show these statements.

(c) How to regulate the step size? For the error (1) in the nth step, (so at time t_{n+1}) it holds

$$\|\mathrm{err}_{n+1}\| = C_n h_n^{s+1} \quad \text{(why?)}.$$

Under the sometimes unrealistic assumption $C_{n+1} \approx C_n$ we obtain under an estimate for err_{n+1} and the request that $\|\mathrm{err}_{n+1}\| \approx 1$ the step size for the next step as

$$h_{\mathrm{new}} := \mathrm{fac} \cdot h_{\mathrm{old}} \, \|\mathrm{err}_{n+1}\|^{-1/(s+1)} \tag{2}$$

with the same weighted norm

$$\|\mathrm{err}_{n+1}\| = \sqrt{\frac{1}{d}\sum_{i=1}^{d}\left(\frac{\mathrm{err}_{n+1,i}}{\mathrm{sc}_i}\right)^2}, \qquad \mathrm{sc}_i = \mathrm{Atol}_i + \max\{|y_{n,i}|, |y_{n+1,i}|\}\,\mathrm{Rtol}_i,$$

and a factor fac which is dependent on the maximal number of Newton steps $k_{\max}$ and the number of made Newton iterations Newt in the current Runge–Kutta step. It is given by

$$\mathrm{fac} = 0.9 \cdot \frac{2k_{\max} + 1}{2k_{\max} + \mathrm{Newt}}.$$

Here, Atol_i and Rtol_i are tolerances for the absolute and relative error.

In the case $h_{\mathrm{new}} < \mathrm{fac} \cdot h_{\mathrm{old}}$ it follows $\|\mathrm{err}_{n+1}\| > 1$ (?), i.e. a step size reduction of more than fac is not possible without rejection of the step.

(d) A realistic assumption is $C_{n+1}/C_n \approx C_n/C_{n-1}$. Show that from $C_{n+1}h_{\mathrm{new}}^{s+1} = 1$ it follows for the new step size

$$h_{\mathrm{new}} := \mathrm{fac} \cdot h_n \left(\frac{1}{\|\mathrm{err}_{n+1}\|}\right)^{1/(s+1)} \cdot \frac{h_n}{h_{n-1}} \left(\frac{\|\mathrm{err}_n\|}{\|\mathrm{err}_{n+1}\|}\right)^{1/(s+1)}. \tag{3}$$

A possible step size strategy lies for example in the choice of the minimum from (2) and (3).

Solutions are discussed on Tuesday June 17, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.