Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

Tübingen, June 3, 2026

7. Sheet for Numerics of Instationary Differential Equations

Exercise 17:

Show (with assumptions from the lecture): the solution $u(t) \in V$ of the homogeneous parabolic initial boundary value problem $u' + Au = 0$ in V', $u(0+) = u_0$ in H, satisfies for all $t > 0$

$$Au(t) \in H \quad \text{and} \quad |Au(t)| \leq \frac{C_1}{t} |u_0|$$

and thus

$$\|u(t)\| \leq \frac{C_2}{\sqrt{t}} |u_0|,$$

where the constants C_1, C_2 are independent of t and u_0.

Exercise 18:

Show that, under the assumptions of exercise 17

$$|u^{(k)}(t)| \leq \frac{C_k}{t^k} |u_0|, \qquad \text{for } t > 0 \text{ and } k \geq 1.$$

Exercise 19: (Crank-Nicolson method)
Discretizing a parabolic problem using finite elements in space and the midpoint rule in time yields the following scheme:

For $n = 0, 1, 2, \dots$, find $u_{n+1} \in V_h$ such that

$$\Big((u_{n+1} - u_n)/\tau, v \Big) + a\Big((u_{n+1} + u_n)/2, v \Big) = \Big(f\big((t_{n+1} + t_n)/2 \big), v \Big), \qquad \text{for all } \ v \in V_h \,.$$

(a) In each step, this is a linear equation system in $\mathbb{R}^N$. Derive this system.

(b) Derive a stability estimate using energy equations.

Exercise 20: (Crank-Nicolson method)

Show that, in the situation of the previous exercise and under suitable regularity assumptions, the following error estimates hold for $n\tau \leq T$

$$|u_n - u(t_n)| \leq C\,(h^2 + \tau^2),$$

$$\left(\tau \sum_{j=0}^{n-1} \left\| \frac{u_{j+1} + u_j}{2} - u\Big(\frac{t_{j+1} + t_j}{2} \Big) \right\|^2 \right)^{1/2} \leq C\,(h + \tau^2)\,.$$

Solutions are discussed on Tuesday June 10, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.