

4. Sheet for Numerics of Instationary Differential Equations

Exercise 10: Consider the parabolic differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) - a_0(x)u && \text{in } \Omega \times (0, T), \\ u &= 0 && \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega \times \{0\}, \end{aligned}$$

where Ω is a given bounded domain in \mathbb{R}^d with piecewise continuously differentiable boundary Γ . The coefficient functions $a_{ij}, a_0 : \bar{\Omega} \rightarrow \mathbb{R}$ are continuous and satisfy,

$$\exists \alpha_0 \geq 0 : \forall x \in \Omega : a_0(x) > \alpha_0,$$

and the matrices $(a_{ij}(x))_{ij}$ are symmetric and on Ω uniformly positive definite, that is

$$\exists \alpha_1 > 0 : \forall \xi \in \mathbb{R}^d, \forall x \in \Omega : \sum_{i,j=1}^d \xi_i \xi_j a_{ij}(x) \geq \alpha_1 \xi^T \xi.$$

Derive the weak formulation of the problem and prove that classical solutions are also weak solutions.

Exercise 11: Let $A \in \mathbb{C}^{N \times N}$. Show: If the eigenvalues of A are inside a circle Γ , then

$$e^{-tA} = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda I + A)^{-1} d\lambda.$$

Hint: Use the Jordan normal form and the fact that each Jordan block is of the form $J = \mu I + N$, where N is nilpotent. You might also need a Neumann series.

Exercise 12:

Consider the heat equation $\partial u / \partial t = \Delta u$ in $\Omega \times (0, T)$ with homogeneous Neumann boundary conditions $\partial u / \partial n = 0$ on $\Gamma \times (0, T)$ and initial value $u(\cdot, 0) = u_0$.

- (a) Derive the weak formulation. What is the space V and the corresponding bilinear form a on V ? Is a V -elliptic?
- (b) Show the Gårding inequality

$$a(v, v) \geq \alpha \|v\|^2 - c|v|^2, \quad \text{for all } v \in V,$$

with $\alpha > 0$, $c \geq 0$. Here $\|\cdot\|$ is the norm of V and $|\cdot|$ the norm of $H = L^2(\Omega)$. Which values of α and c do you get?

Solutions are discussed on Wednesday May 13, 2026

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