Universität Tübingen
Mathematisches Institut
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3. Sheet for Numerics of Instationary Differential Equations

Exercise 7: Consider the differential equation

$$y' = Ay + g(t, y),$$

where

$$\langle Av, v\rangle \le \mu \|v\|^2, \quad \text{for all } v \in \mathbb{R}^d,$$

and g satisfies a Lipschitz condition with constant L.

We apply a linearly implicit Euler-method

$$y_{n+1} = y_n + h\big(Ay_{n+1} + g(t_n, y_n)\big).$$

Prove: If $\mu + L \le 0$, then both the differential equation and the method are contractive.

Exercise 8: Is the following implicit Runge–Kutta method contractive?

$$\begin{array}{c|cc} 0 & & \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

Exercise 9: (Stopping criteria for the Newton iteration, evaluation of the Jacobian)

(a) The simplified Newton method usually converges linearly:

$$\|\Delta Z^{(k+1)}\| \le \bar{\theta}\, \|\Delta Z^{(k)}\|$$

with a θ, that hopefully satisfies $\theta < 1$. Show that in this case, the error after the $(k+1)$-th step satisfies

$$\|Z^{(k+1)} - Z\| \le \frac{\theta}{1-\theta}\, \|\Delta Z^{(k)}\|.$$

Hint: telescope sum for $Z^{k+1} - Z^{k+1+j}$.

(b) One can estimate θ by $\theta_k = \|\Delta Z^{(k)}\| / \|\Delta Z^{(k-1)}\|$. Since the iteration error should not be greater than the local error, which should be $\approx$ tol, one stops the Newton iteration, if

$$\eta_k\, \|\Delta Z^{(k)}\| \le \kappa\, \text{tol}, \qquad \eta_k = \frac{\theta_k}{1-\theta_k}.$$

This strategy can only be applied after at least two iterations. In order to make it possible to stop after the first iteration already, one uses $\eta_0 = \max\{\eta_{\text{old}}, \text{eps}\}$, where eps is the machine accuracy. A good choice for is $\kappa \in [0.01, 0.1]$ (resulting from numerical tests).

To improve efficiency, we limit the number of Newton iterations to $k_{\max} \in \{7, 8, 9, 10\}$. During these $k_{\max}$ steps, the computation is canceled and the step size τ is decreased (e.g. to $\tau/2$) if there exists a k with $\theta_k \ge 1$ (divergence), or if

$$\frac{\theta_k^{k_{\max}-k}}{1-\theta_k}\, \|\Delta Z^{(k)}\| > \kappa \cdot \text{tol}.$$

Think about why the left-hand side of this expression is a coarse estimate for the error after $k_{\max}$ iterations.

If convergence occurs after one step or if the last θ_k is very small, e.g. $\theta_k < 10^{-3}$, then you don't compute a new Jacobian in the next step but continue using the current one.

Solutions are discussed on Wednesday May 6, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.