Universität Tübingen
Mathematisches Institut
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Tübingen, April 22, 2026

## 2. Sheet for Numerics of Instationary Differential Equations

**Exercise 4:**

Show that the stability function of a Runge-Kutta method satisfies

$$R(z) = \frac{\det\big(I + z(\mathbf{1}b^\top - A)\big)}{\det(I - zA)},$$

where $\mathbf{1} = (1, \dots, 1)^\top$.

*Hint:* Use (but don't prove) $\det(I + wv^\top) = 1 + v^\top w$.

**Exercise 5:**

Compute the stability function $R(z)$ of the following Runge-Kutta method (Lobatto IIIC):

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Show that this method is A-stable.

**Exercise 6:**

Consider a collocation method with symmetrically distributed nodes: $c_i = 1 - c_{s+1-i}$ for $i = 1, \dots, s$. Prove that the stability function of this method satisfies

$$R(z) \cdot R(-z) = 1$$

for all $z \in \mathbb{C}$ (except of the poles).

In particular, $|R(z)| \equiv 1$ on the imaginary axis $i\mathbb{R}$.

*Hints:* Let $A$ and $b$ be the coefficient matrix and the coefficient vector of the Runge–Kutta method, respectively, and $\mathbf{1} = (1, \dots, 1)^\top$. Show and use

$$b = Pb, \qquad A = \mathbf{1}b^\top - PAP,$$

where

$$P = \begin{pmatrix} 0 & \cdots & & 0 & 1 \\ 0 & & & \iddots & 0 \\ & & 1 & & \\ 0 & \iddots & & & 0 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}.$$

Then, use exercise 4.

**Solutions are discussed on Tuesday April 29, 2026**
**Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.**