Universität Tübingen
Mathematisches Institut
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Tübingen, July 10th, 2024

Exercise sheet no. 10 – Numerics for instationary differential equations

Exercise 25:

Consider the Crank Nicholson method applied to the heat equation $u_t = u_{xx}$

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{2}\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + \frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{h^2}\right).$$

(a) Derive the method.

(b) Determine the growth factor

$$G(\alpha) = \frac{P(e^{ih\alpha})}{Q(e^{ih\alpha})}$$

(c) Study the stability, that is, determine a γ independent of h, τ and α such that

$$|G(\alpha)| \leq e^{\gamma\tau}.$$

(d) Find a p such that

$$|G(\alpha) - e^{-\tau\alpha^2}| \leq C\tau h^p(1 + |\alpha|^q),$$

where C and q are independent of h, τ and α.

Exercise 26:

As in the previous exercise, formulate the Crank-Nicholson method for the Schrödinger equation $u_t = iu_{xx}$. Study the growth factor $G(\alpha)$, stability and order of the method. In addition, show the conservation of the ℓ^2-norm:

$$\sum_{j=-\infty}^{\infty} |u_j^{n+1}|^2 = \sum_{j=-\infty}^{\infty} |u_j^n|^2.$$

Exercise 27:

Is the initial value problem ($x \in \mathbb{R},\ t \geq 0$)

$$u_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_x + Bu, \quad u_{|t=0} = u_0$$

with a constant matrix $B \in \mathbb{R}^{2\times 2}$ well-posed?

Solutions are discussed on 19.07.2024.
Contact person: Dominik Sulz - `dominik.sulz@uni-tuebingen.de`. Open door policy - just come to my office if you have any questions!