Universität Tübingen
Mathematisches Institut
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Exercise sheet no. 4 – Numerics for instationary differential equations

Exercise 10:

Consider the parabolic differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) - a_0(x)u && \text{in } \Omega \times (0,T) \\ u &= 0 && \text{on } \Gamma \times (0,T) \\ u &= u_0 && \text{in } \Omega \times \{0\}, \end{aligned}$$

where Ω is a given bounded domain in $\mathbb{R}^d$ with piecewise continuously differentiable boundary Γ. The coefficient functions $a_{ij}, a_0 : \overline{\Omega} \to \mathbb{R}$ are continuous and satisfy,

$$\exists\, \alpha_0 \geq 0 : \forall x \in \Omega : a_0(x) > \alpha_0,$$

and the matrices $(a_{ij}(x))_{ij}$ are symmetric and on Ω uniformly positive definite, that is

$$\exists\, \alpha_1 > 0 : \forall \xi \in \mathbb{R}^d, \forall x \in \Omega : \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(x) \geq \alpha_1 \xi^T \xi.$$

Derive the weak formulation of the problem and prove that classical solutions are also weak solutions.

Exercise 11:

Let $A \in \mathbb{C}^{N\times N}$. Show: If the eigenvalues of A are inside a circle Γ, then

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I + A)^{-1}\, d\lambda.$$

Hint: Use the Jordan normal form and the fact that each Jordan block is of the form $J = \mu I + N$, where N is nilpotent. You might also need a Neumann series.

Exercise 12:

Consider the heat equation $\partial u/\partial t = \Delta u$ in $\Omega \times (0,T)$ with homogeneous Neumann boundary conditions $\partial u/\partial n = 0$ on $\Gamma \times (0,T)$ and initial value $u(\cdot, 0) = u_0$.

(a) Derive the weak formulation. What is the space V and the corresponding bilinear form a on V? Is a V-elliptic?

(b) Show the *Gårding inequality*

$$a(v,v) \geq \alpha\|v\|^2 - c|v|^2, \qquad \text{for all } v \in V,$$

with $\alpha > 0$, $c \geq 0$. Here $\|\cdot\|$ is the norm of V and $|\cdot|$ the norm of $H = L^2(\Omega)$. Which values of α and c do you get?

Solutions are discussed on 17.05.2024.
Contact person: Dominik Sulz - `dominik.sulz@uni-tuebingen.de`. Open door policy - just come to my office if you have any questions!