Universität Tübingen
Mathematisches Institut
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Exercise sheet no. 2 – Numerics for instationary differential equations

Exercise 4: Show that the stability function of a Runge-Kutta method satisfies

$$R(z) = \frac{\det\left(I + z(\mathbf{1}b^T - A)\right)}{\det(I - zA)},$$

where $\mathbf{1} = (1, \ldots, 1)^T$.
Hint: Use (but don't prove) $\det(I + wv^T) = 1 + v^T w$.

Exercise 5:

Compute the stability function $R(z)$ of the following Runge-Kutta method (Lobatto IIIC):

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Show that this method is A-stable.

Exercise 6: Consider a collocation method with symmetrically distributed nodes: $c_i = 1 - c_{s+1-i}$ for $i = 1, \ldots, s$. Prove that the stability function of this method satisfied

$$R(z) \cdot R(-z) = 1 \qquad \text{for all } z \in \mathbb{C} \quad \text{(except of the poles)} .$$

In particular, $|R(z)| \equiv 1$ on the imaginary axis $\mathrm{i}\mathbb{R}$.
Hints: Let A and b be the coefficient matrix and the coefficient vector of the Runge–Kutta method, respectively, and $\mathbf{1} = (1, \ldots, 1)^T$. Show and use

$$b = Pb, \qquad\qquad A = \mathbf{1}b^T - PAP,$$

where

$$P = \begin{pmatrix} 0 & \ldots & & 0 & 1 \\ 0 & & & \iddots & 0 \\ & & 1 & & \\ 0 & \iddots & & & 0 \\ 1 & 0 & \ldots & & 0 \end{pmatrix}.$$

Then, use exercise 4.

Solutions are discussed on ??.04.2024.
Contact person: Dominik Sulz - `dominik.sulz@uni-tuebingen.de`. Open door policy - just come to my office if you have any questions!