Universität Tübingen
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Exercise sheet no. 1 – Numerics for instationary differential equations

Exercise 1:

Show that Runge–Kutta- and multistep methods are invariant under linear transformations $y = Tz$. That is, if the method is applied to $y' = f(t, y)$ and $z' = T^{-1}f(t, Tz)$ with initial values

$$y_0 = Tz_0 \quad \text{(RKM)}, \qquad\qquad y_j = Tz_j, \quad j = 0, \ldots, k-1 \quad \text{(MSM)},$$

then we have $y_1 = Tz_1$ resp. $y_{n+k} = Tz_{n+k}$.

Exercise 2:

Show that an explicit s-step Runge–Kutta-Method of order $p = s$ has the stability function

$$R(z) = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^s}{s!}.$$

Therefore, R is independent of the coefficients a_{ij}, b_j, c_j of the Runge–Kutta-Method.

Exercise 3:

Space discretization of the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)\ ,\ u(0,t) = u(1,t) = 0\ ,\ u(x,0) = u_0(x) \quad (0 \le x \le 1\ ,\ t \ge 0)$$

using finite differences yields the following system of ordinary differential equations:

$$y' = Ay\ , \quad y(0) = y_0$$

where

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}, \qquad (N+1)h = 1\ .$$

(a) Compute eigenvalues and eigenvectors of A. What happens with the eigenvalues as $h \to 0$?

(b) How large can the time step size τ be chosen such that the explicit Euler method remains stable? What happens, if larger step sizes are chosen? Answer the same questions with regard to the implicit Euler method.

Hint for (a): An eigenvector $v = (v_1, \ldots, v_N)^T$ corresponding to the eigenvalue λ of A satisfies

$$v_{n-1} - (2 + \lambda h^2)\cdot v_n + v_{n+1} = 0, \qquad (n = 1, \ldots, N),$$

where $v_0 = v_{N+1} = 0$. Therefore (why?), v_n is a linear combination of the n-th powers of the zeros $z_{1,2}$ of the characteristic equation $z^2 - (2 + \lambda h^2)z + 1 = 0$. Remind Vieta's formulas.

(Result: $\lambda_k \cdot h^2 = -2 + 2\cos\frac{k\pi}{N+1}$, $k = 1, \ldots, N$)

Solutions are discussed on 24.04.2024.

Contact person: Dominik Sulz - `dominik.sulz@uni-tuebingen.de`. Open door policy - just come to my office if you have any questions!