Universität Tübingen
Mathematisches Institut
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Tübingen, June 1st, 2018

6th Exercise sheet – Numerics for instationary differential equations

Exercise 15:

Consider the heat equation $\partial u/\partial t = \Delta u$ in $\Omega \times (0,T)$ with homogeneous Neumann boundary conditions $\partial u/\partial n = 0$ on $\Gamma \times (0,T)$ and initial value $u(\cdot,0) = u_0$.

(a) Derive the weak formulation. What is the space V and the corresponding bilinear form a on V? Is a V-elliptic?

(b) Show the *Gårding inequality*

$$a(v,v) \geq \alpha \|v\|^2 - c|v|^2, \qquad \text{for all } v \in V,$$

with $\alpha > 0$, $c \geq 0$. Here $\|\cdot\|$ is the norm of V and $|\cdot|$ the norm of $H = L^2(\Omega)$. Which values of α and c do you get?

Exercise 16:

Show: If the bilinear form in the weak formulation of a initial boundary value problem satisfies just the Gårding inequality (see previous exercise) instead of being V-elliptic, all existence and uniqueness properties of the lecture are still true. The estimates for the solution still hold, with a factor e^{ct} on the right-hand side.

Hint: Formulate an equivalent problem for $w(x,t) = e^{-ct}u(x,t)$ and consider the corresponding weak formulation.

Exercise 17:

Show (with assumptions from the lecture): the solution $u(t) \in V$ of the homogeneous parabolic initial boundary value problem $u' + Au = 0$ in V', $u(0+) = u_0$ in H, satisfies for all $t > 0$

$$Au(t) \in H \quad \text{and} \quad |Au(t)| \leq \frac{C_1}{t}|u_0|$$

and thus

$$\|u(t)\| \leq \frac{C_2}{\sqrt{t}}|u_0|,$$

where the constants C_1, C_2 are independent of t and u_0.

Programming exercise 3 :

Consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t) & \text{in } \Omega \times [0,T] \\ u(x,t) = 0 & \text{on } \Gamma \times [0,T] \\ u(x,0) = u_0(x) & \in \Omega, \end{cases}$$

where Ω is the unit circle. Implement linear finite elements and BDF 2 with constant step size in MATLAB. We choose $u(x,t) = e^{-t}(1 - \|x\|^2)$ as exact solution and thus $f(x,t) = -e^{-t}(1 - \|x\|^2) + 4e^{-t}$.

Groundwork for this exercise is available on the web page. The second initial value is computed with the exact solution. The load vector $\mathbf{b}(t)$ can be approximated by

$$\mathbf{b}(t)_j = \int_\Omega f(x,t)\varphi_j dx \approx \int_\Omega I_h f(x,t)\varphi_j dx = (\mathbf{M}\mathbf{f}(t))_j \,,$$

where $\mathbf{f}(t)$ is the vector of nodal values.

Compute solutions using the grid sizes and time step sizes given in the pseudo code. Generate four plots: The L^2-errors and H^1-errors of the approximation to $u(x,1)$, plotted against h and τ, respectively. The errors are given by $\sqrt{\mathbf{e}^T\mathbf{M}\mathbf{e}}$ and $\sqrt{\mathbf{e}^T\mathbf{A}\mathbf{e}}$, respectively. Explain how the plots correspond to the convergence theory of parabolic equations.

What is the computational bulk of the whole method?

How many lines in the codes have to be changed, if for example, the implicit Euler method is used?

Hints:

- The *grid size* given as input parameter to `distmesh` is not the *real* grid size of the resulting grid, so you have to compute the *real* grid size after the grid is generated.
- Use sparse matrices!
- You may have to avoid the smallest value of h, depending on your computer's performance and memory (overall computation time may be around 5 minutes).
- Feel free to come to my office if something is not clear.

Solutions are discussed on June 8th.
The programming exercise needs to be handed in by June 15th.

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