Universität Tübingen
Mathematisches Institut
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5th Exercise sheet – Numerics for instationary differential equations

Exercise 12:

Consider the differential equation

$$u' = C(t)u + d(t), \qquad u(0) = 0 \in \mathbb{R}^d,$$

with a matrix $C(t) \in \mathbb{R}^{d\times d}$, $t \in [0,T]$. Assume that there exists a matrix A and an invertible matrix B such that

$$\begin{aligned} &\|B^{-1}(C(t) - A)\| \le l, && \text{for } 0 \le t \le T,\\ &\|(\lambda I - A)^{-1} B\| \le m, && \text{for } \Re(\lambda) \ge c, \end{aligned}$$

where $\|\cdot\|$ is a matrix norm that corresponds to a scalar product on $\mathbb{R}^d$.
Prove: If $ml < 1$, then the solution u satisfies

$$\left(\int_0^T \left\|e^{-ct}u(t)\right\|^2\, dt\right)^{1/2} \le \frac{m}{1-ml}\left(\int_0^T \left\|e^{-ct}B^{-1}d(t)\right\|^2\, dt\right)^{1/2}.$$

Exercise 13:

Consider the parabolic differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) - a_0(x)u && \text{in } \Omega \times (0,T)\\ u &= 0 && \text{on } \Gamma \times (0,T)\\ u &= u_0 && \text{in } \Omega \times \{0\}\,, \end{aligned}$$

where Ω is a given bounded domain in $\mathbb{R}^d$ with piecewise continuously differentiable boundary Γ. The coefficient functions $a_{ij}, a_0 : \overline{\Omega} \to \mathbb{R}$ are continuous and satisfy,

$$\exists\, \alpha_0 \ge 0 : \forall x \in \Omega : a_0(x) > \alpha_0,$$

and the matrices $(a_{ij}(x))_{ij}$ are symmetric and on Ω uniformly positive definite, that is

$$\exists\, \alpha_1 > 0 : \forall \xi \in \mathbb{R}^d, \forall x \in \Omega : \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(x) \ge \alpha_1 \xi^T \xi.$$

Derive the weak formulation of the problem and prove that classical solutions are also weak solutions.

Exercise 14:

Let $A \in \mathbb{C}^{N\times N}$. Show: If the eigenvalues of A are inside a circle Γ, then

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I + A)^{-1}\, d\lambda.$$

Hint: Use the Jordan normal form and the fact that each Jordan block is of the form $J = \mu I + N$, where N is nilpotent. You might also need a Neumann series.

Solutions are discussed on June 1st.

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