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7. Exercise sheet for numerics of stationary differential equations

Exercise 17:

Consider the Helmholtz-equation with Neumann boundary conditions:

$$-\Delta u + u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma. \tag{$**$}$$

Show for $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ that the following statements are equivalent:

(a) u is solution to $(**)$

(b) It holds
$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + uv \right) d(x,y) = \int_\Omega fv \, d(x,y) + \int_\Gamma gv \, d\sigma$$
for all $v \in C^1(\Omega) \cap C(\bar{\Omega})$.

(c) u is solution to the variational problem
$$\frac{1}{2}\int_\Omega \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + v^2 \right] d(x,y) - \int_\Omega fv \, d(x,y) - \int_\Gamma gv \, d\sigma = \text{min!}$$
under all $v \in C^1(\Omega) \cap C(\bar{\Omega})$.

Exercise 18:

We define: $u \in L^2(\Omega)$ admits the *weak derivative* $\partial_i u$ (for $i = 1, \dots, n$), if $\partial_i u \in L^2(\Omega)$ and

$$(\phi, \partial_i u)_0 = -(\frac{\partial \phi}{\partial x_i}, u)_0 \qquad \text{für alle } \ \phi \in C_0^\infty(\overline{\Omega}) \ .$$

Show for bounded piecewise C^1 domains Ω:

(a) For $u \in C^1(\overline{\Omega})$ is the classical derivative $\partial u / \partial x_i$ also a weak derviative.

(b) For $u \in H^1(\Omega)$ are the generalized derivatives (from the lecture) also weak derivatives.

It holds (you do not need to proof that): If the weak derivatives of $u \in L^2(\Omega)$ exist, they are also generalized derivatives and therefore $u \in H^1(\Omega)$.

Exercise 19:

Let V, W be normed vector spaces and $L : V \to W$ a linear map. Show:

$$L \text{ continuous} \iff L \text{ continuous in } 0 \iff L \text{ bounded.}$$

Discussion of the sheet on 04.12.2023.