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5. Exercise sheet for numerics of stationary differential equations

Exercise 10:

Given is the parameter-dependent differential equation

$$y'(t) = f(t, y(t), p).$$

Show that

$$P(t) = \frac{\partial}{\partial p} y(t|t_0, x_0, p)$$

is the solution of the inhomogeneous, linear matrix differential equation

$$P'(t) = C(t)P(t) + B(t),$$

where $C(t) = \frac{\partial f}{\partial y}(t, y(t|t_0, x_0, p), p)$ and $B(t) = \frac{\partial f}{\partial p}(t, y(t|t_0, x_0, p), p)$ with $P(t_0) = 0$.

Exercise 11:

If one approximates in the variational problem

$$(\star) \qquad \int_a^b f(t, y, y')dt = \text{min!}$$

the integral through the trapezoidal rule and the derivatives through a difference quotient, one gets to the following minimization problem

$$\frac{h}{2} f\left(a, y(a), \frac{y(a+h) - y(a)}{h}\right) + h \sum_{i=1}^{n-1} f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) + \frac{h}{2} f\left(b, y(b), \frac{y(b) - y(b-h)}{h}\right) = \text{min!}$$

Show for given boundary values that the solution of this problem is the same as if one applies the midpoint rule to the Euler Lagrange equations

$$y' = v, \quad p' = \frac{\partial f}{\partial y}(t, y, v), \quad 0 = p - \frac{\partial f}{\partial y'}(t, y, v).$$

Hint: The derivative in the midpoint rule is replaced by a difference quotient of the form

$$y'(t_i) \to \frac{y_{i+1} - y_{i-1}}{2h}$$

(analogous for p').

Exercise 12:

Formulate and prove a 2-dimensional version of the fundamental lemma of calculus of variations.

Exercise 13:

Derive the Euler Lagrange equations for the minimal surface problem

$$\int_\Omega \sqrt{1 + u_x^2(x,y) + u_y^2(x,y)}\, d(x,y) = \text{min!}$$

with given values on the boundary $u = g$ on $\partial\Omega$. Here Ω is a bounded domain with smooth boundary in $\mathbb{R}^2$.

Discussion of the sheet on 20.11.2023.