Universität Tübingen
Mathematical Institut
Prof. Dr. Christian Lubich

Tübingen, the 23.10.2023

2. Exercise sheet for numerics of stationary differential equations

Exercise 3:

For the boundary value problem

$$y' = C(t)y + q(t), \qquad Ay(a) + By(b) = 0$$

consider the sensitivity matrix

$$E(t) = AR(a,t) + BR(b,t) \qquad (a \le t \le b).$$

(a) Show: $E(t)$ is invertible for all $t \in [a,b] \iff E(t)$ is invertible for one $t \in [a,b]$. This is fulfilled in the following.

(b) Show: The unique solution of the above boundary value problem is given by given by

$$y(t) = \int_a^b G(t,s)q(s)ds$$

with the Greens function

$$G(t,s) = \begin{cases} E(t)^{-1}AR(a,s) & \text{für } a \le s \le t \le b \\ -E(t)^{-1}BR(b,s) & \text{für } a \le t \le s \le b. \end{cases}$$

Hint: Represent $y(t)$ as the sum of the solution $v(t)$ of the associated initial value problem with initial value $v(a) = 0$ and the solution $w(t)$ of the associated homogeneous initial value problem with suitable initial value $w(0) = w_0$.

(c) (Sensitivity against perturbations of the inhomogeneity)
Let $y, \tilde{y}$ be the solutions of the boundary value problem

$$\begin{aligned} y' &= C(t)y + q(t), \qquad & Ay(a) + By(b) &= r \\ \tilde{y}' &= C(t)\tilde{y} + \tilde{q}(t), \qquad & A\tilde{y}(a) + B\tilde{y}(b) &= r. \end{aligned}$$

Show:

$$\max_{a \le t \le b} \|y(t) - \tilde{y}(t)\| \le \gamma \max_{a \le t \le b} \|q(t) - \tilde{q}(t)\|$$

with $\gamma = \max_{a \le t \le b} \int_a^b \|G(t,s)\|\, ds \le (b-a) \max_{a \le s,t \le b} \|G(t,s)\|$

Exercise 4:

(a) Reformulate the initial value problem (with real parameter $\lambda \neq 0$)

$$u'' = \lambda^2 u, \qquad u(0) = 0, \quad u(1) = 1$$

into a 1st order system by introducing $v = u'/\lambda$. Calculate its resolvent and the Green's function of the boundary value problem. Prove that for $\lambda \to +\infty$ the resolvent grows like e^λ, whereas the Green's function remains limited independently of λ.

(I.e. the initial value problem is ill conditioned, the boundary value problem is well conditioned.)

(b) For which values of $\omega \in \mathbb{R}$ is the boundary value problem

$$u'' = -\omega^2 u, \qquad u(0) = 0, \quad u(1) = 1$$

uniquely solvable? How do the resolvent of the initial value problem and the Green's function of the boundary value problem behave for $\omega \to \pi$?

(Initial value problem well-conditioned, boundary value problem ill-conditioned)

Hints: $R(t,s) = e^{C(t-s)}$, diagonalize C. $\lambda = i\omega$ in (b) saves computational effort.

Programming exercise 1 :

Implement the single shooting method for the boundary value problem

$$\begin{aligned} u''(t) &= \lambda \cdot (u(t))^2, \qquad t \in [a,b], \\ u(a) &= u_a, u(b) = u_b, \end{aligned}$$

with $\lambda \in \mathbb{R}$. Follow the steps:

- Implement a classical Runge-Kutta method to solve the corresponding initial value problem in each Newton step (as well as to solve the initial value problems to determine the resolvent needed in the Newton method) the classical Runge-Kutta method.
- Calculate the matrices A^k, B^k, and C^k defined in the lecture by hand.
- To solve the linear equation system in each Newton step you can use the built-in Matlab operator \.
- Stop as soon as the boundary conditions are fulfilled up tp a tolerance error $< TOL$.

Test your program with $a = 0, b = 1$, $u_a = 0$, $u_b = 1$, $\lambda = 1/2$ and $TOL = 1e-7$. Use as initial value $u'(0) = -5$. Plot your trajectories of the found approximation to the solution against time t. Compare it with the trajectory of the corresponding initial value problem $u(0) = u_a, u'(0) = s$ with $s = -4, -12$.

Discussion of the sheet on 30.10.2023.
Hand in the programming exercise by 06.11.2023.