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13. Exercise sheet for numerics of stationary differential equations

Exercise 36:

Consider the eigenvalue problem

$$\begin{aligned} -u''(x) &= \lambda u(x) \qquad \text{in } (0,\pi), \\ u(0) &= u(\pi) = 0. \end{aligned}$$

Linear finite elements with an equidistant partitioning of $[0,\pi]$ in $(N+1)$ sub-intervals of the length h are chosen for discretisation.

(a) Give the matrices A and M of the corresponding linear equation system $A\mu = \lambda M\mu$.

(b) The continuous eigenvalues and -vectors are

$$\lambda_m = m^2, \qquad u_m(x) = \sin(mx), \qquad m = 1,2,\ldots$$

The discrete eigenvalues and -vectors are

$$\lambda_{m,h} = \frac{6}{h^2}\frac{1-\cos mh}{2+\cos mh}, \qquad u_{m,h} = \Pi_h u_m, \qquad m = 1,2,\ldots,N.$$

Here. $\Pi_h v = \sum_{n=1}^N v(nh)\varphi_n$ denotes the piecewise linear interpolation in the grid points with finite element basis $(\varphi_n)_{n=1}^N$. Verify this and show further the estimate

$$\lambda_m - \lambda_{m,h} \leq C(m)h^2.$$

Exercise 37:

Consider the initial/boundary value problem of the heat equation

$$\begin{aligned} \frac{\partial}{\partial t}u - \Delta u &= 0 && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u &= u_0 && \text{for} t = 0 \end{aligned}$$

with $u_0 : \overline{\Omega} \to \mathbb{R}$. Show:

(a) If a classical solution $u : \overline{\Omega} \times [0,T]$ exists, it is given by

$$u(\cdot,t) = \sum_{m=1}^{\infty} (u_0, w_m) e^{-\lambda_m t} w_m,$$

where λ_m and w_m are the eigenvalues and $L_2(\Omega)$-orthogonal eigenvectors of the Laplace-operators.
<u>Hint:</u> Use the ansatz $u(x,t) = \varphi(t)w(x)$.

(b) Derive from (a) for all $k \in \mathbb{N}$:

$$\left\| \frac{\partial^k}{\partial t^k} u(\cdot,t) \right\|_0 \leq C(t,k)\|u_0\|_0.$$

What happens in the case $t \to \infty$?

(c) Propose a numerical method or solving this problem.
<u>Hint:</u> Part (c) can be substituted by visiting the lecture numerics of instationary differential equations in the summer semester 2024.

Discussion of the sheet on 29.01.2024.