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12. Exercise sheet for numerics of stationary differential equations

Exercise 33:

Let $A = M - N$ by a decomposition of the symmetric, positive definite matrix A and assume N being also symmetric, positive definite. Show that the iteration

$$x_{k+1} = x_k + M^{-1}(b - Ax_k)$$

converges and that the eigenvalues of the iteration matrix are real and lie between 0 and 1.

Exercise 34:

The iteration in the two-grid method can be written in the from

$$u_h^{(k+1)} = Mu_h^{(k)} + v_h$$

with $v_h := (I - M)u_h$.

Give the matrix M explicitly for the case where one uses ν_1 smoothing steps before and ν_2 smoothing steps after each iterations. Further, show that the spectral radius of M only depends on the sum $\nu_1 + \nu_2$ an not how many smoothing steps are done a priori and how many a posteriori.

Exercise 35:

Let H be a Hilbert space and $T : H \to H$ compact. Show that if $\dim H = \infty$ then $0 \in \sigma(T)$.
Hint: $\sigma(T)$ is the spectrum of a linear operator and defined via $\sigma(T) := \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ not bijektiv}\}$.

Programming exercise 5 :

Solve the linear equation system from programming exercise 4 using a two-grid (or if you like multi-grid) method with two Gauss-Seidel iterations as smoothing. Use a linear interpolation for the restriction and prolongation.

Alternatively, you can solve the problem

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

on the unit square $\Omega = [0,1]^2$ using a two grid (or if you like multi-grid) method with two Gauss-Seidel iterations as smoothing. Use a linear interpolation for the restriction and prolongation. (Hint: What are the relations between P and R^T?)

For the computation of the discrete Laplace operator you can use the following Matlab function and the (incomplete) script.

```
function A=stiffness_matrix(h)
% A - the stiffness-matrix for the 5-star finite differences approximation
% h - grid size

% number of nodes in one direction which are not on the boundary
% (with boundary n+2 x n+2 Grid)
n=1/h-1;

%% stiffness matrix
% help vector
e=ones(n,1);
% help matrices
D=sparse(4*diag(e));
nD=spdiags([e 0*e e],[-1 0 1], n,n);
id=spdiags(e,[0], n,n);
% stiffness matrix
A=1/h^2 * (kron(id,D-nD) + kron(nD,-id));
```

```
h=1/32; % grid size
n=1/h-1;
N=n^2; % size of the linear equation system

% matrix
A=FD_SteifigkeitsMatrix(h);
% load vector
b=ones(N,1);

% solution of linear equation system
u=A\b;

%% figure without boundary values(!)
figure
surf(reshape(u,n,n))
```

Discussion of the sheet on 22.01.2024.
Hand in the programming exercise by 29.01.2024.